QATAR UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

LINEAR AND BAYESIAN ESTIMATION OF THE PARAMETERS OF THE TYPE II GENERALIZED LOGISTIC DISTRIBUTION BASED ON PROGRESSIVELY TYPE II CENSORED DATA

BY

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A Thesis Submitted to

the College of Arts and Sciences

in Partial Fulfillment of the Requirements for the Degree of

Masters of Science in Applied Statistics

June 2022

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ABSTRACT

RIMAWI, RANA, A., Masters : June : 2022, Applied Statistics

Title:_Linear and Bayesian Estimation of the Parameters of the Type II Generalized Logistic Distribution Based on Progressively Type II Censored Data

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Generalized distributions have become widely used in applications recently. They are very flexible in data analysis, especially with skewed models that are important and occur frequently in many applications. In particular, the Generalized Logistic Distribution with its several types has lately gained a lot of attention in the literature. In this study, based on progressively Type II censored data, we obtained estimators of the unknown parameters of Type II Generalized Logistic Distribution. Several point estimation methods are used. Specifically, we consider the maximum likelihood estimation (MLE), the Bayesian estimator based on importance sampling and Lindley's approximation, the linear estimators (BLUE and BLEE). The estimators were investigated and compared using simulation techniques in a variety of scenarios and progressive censoring schemes. The criteria used for comparison are the mean squared error (MSE) and bias. The derived estimators are applied to real-world data in order to see how they operate in real situations.

Keywords: Point Estimation, Type II Generalized Logistic Distribution, Progressive Censoring

DEDICATION

I devoted this work to my beautiful family especially my husband, daughters and mother. Also, to my friends and my supervisor

ACKNOWLEDGMENTS

This study would not have been possible without the support of many people. Many thanks to my advisor, Prof. Ayman Bakleezi, who supervised me through this journey. Also, thanks to my committee members, Dr. Esam Mahdi, and Dr. Saddam Abassi.

Thanks to the Qatar University for providing all the needs to accomplish this study. Finally, my husband, my daughters, family, and numerous friends who endured this long process with me, always offering support and love.

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CHAPTER1: INTRODUATION

1.1. Introduction

In this chapter we showed a general overview about survival time, censoring and its main types, progressive censoring, generalized distributions and Type II generalized logistic distribution. Then we introduced the literature reviews based on generalized logistic distribution, progressive censoring sampling and generalized logistic distribution under progressive censoring. Also, we introduced the research problem statement, research objective and significant, research specific objectives and scope of the study.

1.2. Overview

Survival time is defined by concepts such as event time, lifespan, and failure time. For variable T, that is a continuous random variable with non-negative values. The real time of this variable indicates the time spent waiting for a well-defined event to occur, i.e., the duration from start time to end time of a considered event for an item (Klein and Moeschberger, 2006). The amount of time spent in a job, the gap between recurrences of symptoms, or the duration of time jobless can all be used to determine the time to event. Time may be represented by fractions of a second, hours, days, months, and even years.

Censoring is a fundamental analytical issue that must be considered in most survival analyses. In essence, censorship happens when we have some knowledge about a person's survival time but not the precise period.

1.2.1. Censoring

By far the most important issue that survival analysis tackles is censorship. When the event of interest does not happen for such individuals before the research finishes, this is referred to as censoring. When a scholar just has a rudimentary awareness of the participants' survival periods and isn't always aware of the exact survival times, this occurs.

We may use the case study to demonstrate the idea. If a researcher is looking at 70 recently enlisted soldiers who have had contact with persons who have tested positive for COVID-19 throughout a 14-day of quarantine and surveillance time. Investigating the incident to see how long it takes for people to show signs of the infection. It's probable that some patients may drop out, that others will pass away due to unrelated conditions, and that just a handful of the patients will exhibit any evidence of infection after the 14-day perception phase but will show signs after the isolation time. The researcher might not even be able to estimate the survival period of the preceding three instances. They only recognize that the period is at least that long, but they have no idea how long it will take them to survive.

As previously stated, survival data usually comprises a response variable which calculates the amount of time up to a certain event happens, as well as a collection of independent variables that are assumed to be linked to the event-time variable.

Event periods include component lifetimes in industrial reliability, work durations, and clinical trial survival times. The goal of survival analysis is to find the model of the theoretical distribution of event timings as well as determine if the event time is dependent on other explanatory variables. Because of a removal or termination of the study, the event time is often not recorded; this is known as censoring. Both censored and uncensored observations are used correctly in survival analysis methods.

There are several different sorts of censorship. For instance: left censoring, interval censoring, and right censoring.

Left Censoring: It is characterized as left censored if a time X linked with a certain participant in research does not exceed the time (a) which is a censoring time, then we can define this time (a) as a left censoring. To put it in another way, the participant must encounter the interested event before being observed in the research at time (a) (T< a), entitle capital T as a random variable for a subject's survival period. Because T stands for time, it may take on any non-negative value; such that, T can be any integer greater than or equal to zero. We believe they saw the occurrence earlier than scheduled (a) for such subjects, but the exact date or time is uncertain. X is observed if and only if, it is not smaller than the precise time (a). We can represent the data that is coming from a left-censored sampling strategy by two random variables (T, δ), where T equals X if the lifetime is observed. d denotes if the lifetime points out to an observed event (δ = 1) or is censored (δ = 0). For left-censoring T = max $(X_i\ ,\ a)$, where i = 1,…,n.

Interval Censoring: The term "interval-censored" refers to a subject that has been censored at regular intervals. When it is observed for some time, then goes missing from follow-up for some time before returning and continuing to be examined. (a < T < b). The observed data in interval-censoring is made up of intervals, with the response falling within the interval for each. An uncensored observation of an observed death in this situation refers to a single point observed interval. Assume we've completed a study on a certain subject, at a certain time in the past (t1), with a negative result from the subject. However, the patient tested positive at a later point in time (t2). We know the subject was exposed to the virus between t1 and t2, but we don't know when. For example, if the time to remission in a clinical trial was properly evaluated, and the (i^{th}) patient was in remission at the 8th week after the trial, but didn’t show up for consecutive review, and then recovered and was no longer in remission at the end of the 11^{th} week, then (i^{th}) patient's censoring interval or remission length is defined by I_i= [8,11).

Right Censoring: If we suppose that there is a time X and a censoring timeframe b for a single subject under investigation, where the X's are distributed independently identically having f(t) and S(t) as probability function and survival function respectively. If X is less than or equal to b, the subject's lifetime X will be defined; if it is higher than b, then the subject is known as a survivor, and his event time will be censored at b. This study's findings may be expressed by couples of random variables. (T, d), where d denotes if the lifetime is associated with an event (δ = 1) or it is censored (δ = 0), we consider T is equal to X when the lifetime is observed otherwise it is equal to b when censored. For right-censoring T = min (X_i , b), Considering T as a time variable and a and b are time points, and i = 1,…, n.

Type I Censoring: When research is expected to be completed at a specific time T determined by the experimenter, Type I censoring occurs. Subject who could not see the event is considered as censored at the conclusion of the research period. In I censorship, the quantity of uncensored items is a random variable.

Type II Censoring: It's possible that the time will be left open at the start of Type II censoring. The experiment is allowed to be executed until a certain percentage of the n objects, r/n, has "failed." The random sample with ordered values T_1, T_2,..., T_n are denoted by T_1, T_2,..., T_n. Our concern is the first r^{th} smallest observations in a n sized random sample, since the observation will end when the r^{th} failure time occurs.

Alternatively, imagine that just the first r < n lifetimes are seen for a given sample T_1, T_2,..., T_n of with n items. Before looking at the survival data, the value of r is set. This signifies that the observed data is made up of the r observations that are the smallest. Other statistics can be used to represent this in terms of random variables.

Only the first r ranked replies $T_1, T_2,..., T_n$ are seen among the available responses $T_1, T_2,..., T_n$. That is to say,

$$t_{(1)} \leq t_{(2)} \leq t_{(3)} \leq ... \leq t_{(r-1)} \leq t_{(r)}$$

Random Censoring: The overall length of observation is fixed in random censoring, although individuals begin the study at various times. Some people are exposed to interested occurrences. Others do not, and several get lost in the shuffle. Some will still be alive after the study is over. The censored objects in random censoring do not all have the same censorship time. When there is just one finishing time, but the entry times vary randomly among the participants, random censoring can be achieved.

Double Censoring: Double censoring is achieved by mixing right and left censoring. In this situation, Z_i =max(min(T_i , t_i), l_i), where l_i , t_i are the right and left censoring times connected with T_i , respectively, and l_i , t_i , in this example, T_i is only observed if it comes inside a window of observations (l_i , t_i),. Otherwise, one of the windows' endpoints is noticed, while the other window's endpoint is most likely hidden. It is also worth noting that double-censoring differs from interval-censoring.

Noninformative Censoring: Participants who drop out of research using noninformative censoring should not act so for any reason unassociated to the research. When there is not information on the distribution of censorship times (C) gathered from the distribution of survival times (T), noninformative censoring happens. That is, the justification for why the circumstance of the occurrence was not recorded had nothing to do with the conclusion under investigation. The research simply came to an end when the people who were being monitored were still alive. The hidden chances of getting the occasion of interest are no different for both uncensored and censored information because of noninformative censoring.

The distinction between informative and not informative censoring is that the censored observations' survival time for acquiring the event of interest is unknown for noninformative, while individuals are lost to follow-up for reasons linked to the study, is known as informative censoring.

Independent Censoring: Independent censoring happens when the units that didn't experience the interest event at time t, are example of all the units in that subgroup who were still at risk, considering their survival experience, at time t. To put it another way, items that have risk value different than the average cannot be censored (removed).

1.2.2. Progressive Censoring

One type of censoring is progressive censoring sampling. The basic idea behind a progressive censoring technique is that it allows units to be removed at each recorded failure time. Recently, the lifetime distribution which created on progressively censored data has received an impressive consideration. Specially to estimate unknown parameters or related survival and hazard functions. The scheme of progressive censoring is of significant value in life-test studies. It permits the elimination of live items from the experiment at diverse levels. This will potentially save plenty for the researcher with respect to cost and time.

Type-II Progressive Censoring: Deletions are carried out during failure periods that have been identified in the progressive Type-II censoring procedure. When a failure is detected, the number of units, that are pre-established, are instantly taken from the remaining units. As a result, the number of observations is predetermined, but the duration of the experiment is unpredictable. The following approach can be used to generate Type-II censored order statistics in a progressive manner. For example, to test the reliability of products, n units are put down at the same time. When the first failure is occurred, R_1 surviving units are randomly selected and removed from the experiment. After the second failure, immediately R_2 units are withdrawn and so on. After the m^{th} failure, the process is repeated until all R_m remaining units are removed. The R_i 's are specified before starting the experiment. The investigator determines number of items in the sample n, the number of failures m, and scheme of the progressive censoring samples. before starting a life experiment. (R_1, R_2 , …, R_m). With n = m $+\sum_{i=1}^{m} Ri$. Figure 1 is an explanation of the Type II progressive censoring sample where the n, m and R are fixed values while the completed time of the experiment is a random variable. (Siyi Chen and Wenhao Gui, 2020)

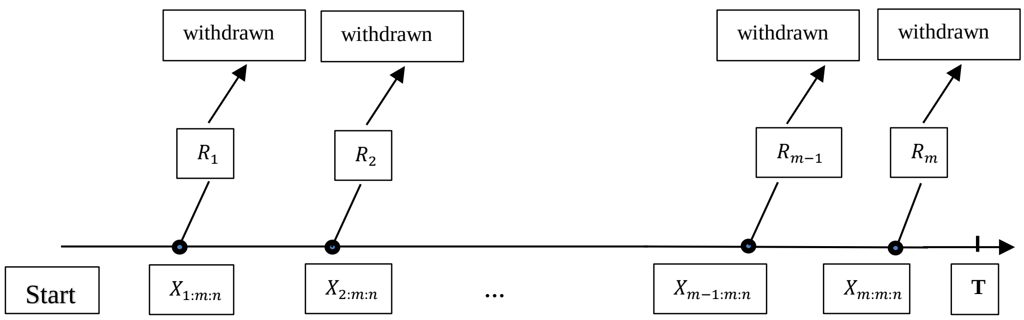

Figure 1: Illustration of Progressive Type-II Censoring

Type-I Progressive Censoring: Type I progressive censoring is based on predetermined time points T_1, T_2,..., T_m. The failure times are recorded one by one till the experiment is completed at time point T_m. As a result, the intervention times T_1, T_2,..., T_m have fixed values, however, the sample size as well as the censoring scheme that was utilized were chosen at random (and may vary from the originally planned censoring scheme at a certain time owing to the lack of enough surviving units to complete the requisite censorship), and the period of the test is constrained by T_m, whereas it is not fixed (random) in the case of progressive Type II censoring.

Figure 2 is an explanation of the type I progressive censoring sample where the n and R are random variables while the completed time of the experiment is a fixed value. (Balakrishnan and Cramer, 2014)

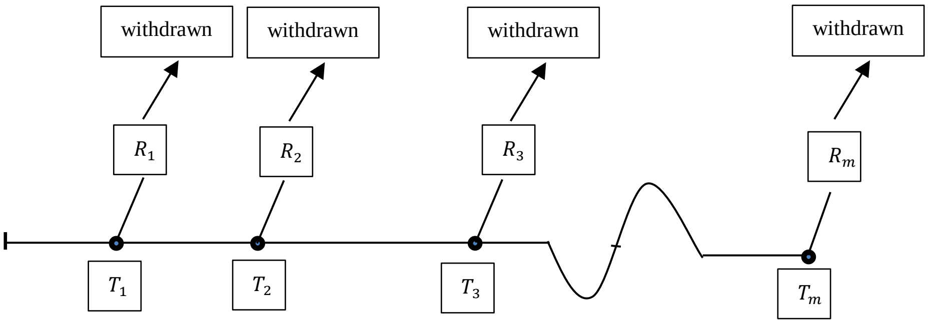

Figure 2: Illustration of Progressive Type-I Censoring

If $R_1 = R_2 = \ldots = R_m = 0$, So, n = m that represents the complete sample.

If $R_1 = R_2 = \ldots = R_{m-1} = 0$, we have R_m= n − m that represents to the conventional Type.

1.2.3. Generalized Distributions

Distributions in statistics are widely applied to explain actual events. The concept of distributions in statistics is being researched in depth, and additional distributions are produced, due to its effectiveness. In the statistics profession, there is still a lot of interest in constructing more adaptable distributions in statistics. There have been several forms of generalized distributions developed and used to diverse phenomena. The fact that such generalized distributions have extra parameters is a typical feature. According to Johnson et al. (1994), four-parameter distributions should suffice for most practical tasks. According to these authors, at least three parameters are required, but they doubt that adding a fifth or sixth parameter will result in a significant improvement.

The induction of location, scale, and shape parameters broadened the traditional continuous probability distributions. Recent efforts have been undertaken to build novel probability distributions that offer the advantage of greater flexibility in fitting specific and multiple real-world sequences of events. The development in G-Classes began with Alzaatreh et al 's, (2013), essential essay, in which they proposed the transformed (T)-transformer (X) (T-X) family.

Assume the random variable $T \in [c,d]$ for cd$-\infty \leq c < d \leq \infty$ with the probability distribution function (pdf) r(t), and a link function, $W(.):[0,1] \rightarrow R$, that meets the following requirements: $W[G(x)] \in [c,d]$ is monotonically non-decreasing and differentiable for any baseline cumulative distribution function (cdf) G(x), for $x \rightarrow -\infty$, $W[G(x)] \rightarrow c$, and for $x \rightarrow \infty$, $W[G(x)] \rightarrow d$. As a result, the T-X class's cdf has the form

$$F(x) = \int_{a}^{W[G(x)]} r(t)dt.$$

Using the T-X method, many authors created extended generalized families. Beta-G, Kw-G type-1, log-gamma-G type-2, gamma-X , exponentiated T-X, Weibull-G, exponentiated-Weibull-H, and generalized odd Lindley-G are some examples of generalized classes.

Among the generalized Distributions, Johnson et al. (1995) describe a Type I generalized logistic distribution as a special form of exponentiated-exponential-logistic distribution. If f(x) is the standard logistic distribution p.d.f., then

$$g(x) = \frac{\alpha\lambda e^{-\lambda x}}{(1+e^{-x})^{\lambda+1}}\left(1-\frac{e^{-\lambda x}}{(1+e^{-x})^{\lambda}}\right)^{\alpha-1} \quad ; \qquad -\propto < x < \propto; \ \alpha,\lambda > 0 \,. \qquad (1)$$

The exponentiated-exponential-logistic distribution becomes a Type I generalized logistic distribution when λ=1 and reduced to standard logistic distribution when α=λ=1. Referring to Balakrishnan and Hossain (2007), if X is a random variable from the probability function Type I Generalized Logistic Distribution, then −X has the probability function Type II Generalized Logistic Distribution. Many authors turned to use these new generalized distributions because of their flexibility, accuracy and ability to be extensible to a new distribution.

1.2.4. Type II Generalized Logistic Distribution

Skewed distributions have lately played a significant role in several research in order to demonstrate extremes. Some of statistical models exist in the literature are used to analyze the lifetime data. The models are chosen in order to present the lifetimes as closely as possible. Among current models, logistic distribution has ended up very famous within the analysis of statistics springing up within the field of biology, actuarial science, industry, and engineering.

The logistic distribution has for quite some time been one of the most successful statistical distributions due to its clarity and traditional relevance as a growth curve. (Erkelens, 1968). Johnson and Kotz (1970) demonstrated how logistic distribution may be used to examine quanta1 response data, probit analysis, and dosage response research, among other things. In a series of investigations, Berkson (1944, 1951, 1953) has demonstrated its use in the context of bioassay. Since the logistic distribution resembles the normal distribution in shape, it is simpler and more lucrative to utilize the logistic distribution instead of the normal probability function to simplify the investigation without introducing too many conflicts between the two theories.

Order statistics from the logistic distribution have been researched in depth by a number of authors; see, for example, Birnbaum and Dudman (1963), Gupta and Shah (1965), Tarter and Clark (1965), Shah (1966, 1970), and Gupta, Qureishi, and Shah (1965, 1967). Tarter (1966) looked at order statistics from truncated logistic distributions, Balakrishnan and Joshi (1983), and Balakrishnan and Kocherlakota (1986). Balakrishnan (1985) carried out a similar study for the half logistic distribution, and Balakrishnan and Puthenpura (1986) demonstrated its usefulness in the field of dependability Lawless (1982) and Mann, Schafer, and Singpurwalla (1974) both mention order statistics from the logistic distribution in the fields of life testing and reliability; see also Hall (1975), who established acceptance sampling plans and tolerance limits under order statistics from the logistic distribution.

Balakrishnan and Leung (2007) presented three forms of Generalized Logistic Distributions. Type I are positively skewed distributions with a greater kurtosis than the logistic; Type II are negatively skewed distributions with a higher kurtosis than the logistic; and Type III are symmetric distributions with a lower kurtosis than the logistic. As a result, the three forms of Generalized Logistic Distributions described in this study will be tremendously useful in research on resilience.

The different three types of Generalized Logistic Distributions are created by combining a double exponential distribution, a reduced log-Weibull distribution, and an exponential-gamma distribution with a gamma distribution.

Hofmann et al. (2005) demonstrated that Type II progressive censoring systems outperform standard Type II censoring methods.

The fact that when $\alpha < 1$, the Generalized Logistic Distribution is negatively skewed and positively skewed when $\alpha > 1$ is one of its features. The Type II Generalized Logistic Distribution is a decreasing function of α. The Type II Generalized Logistic Distribution has 'heavier tails' if the value of goes to infinity, hence it is employed in robustness tests of various standard techniques based on normality or skewed distributions. The following are some additional characteristics and their relationships with other distributions:

1. If $\alpha = 1$ then we'll have usual logistic distribution.
2. If X is a random variable from Type I Generalized Logistic Distribution, then −X has a Type II Generalized Logistic Distribution. As a result, their attributes are identical.
3. The moment-generating function (MGF) may be used to produce moments, allowing us to calculate the mean and variance.
4. Applying the standard Type II Generalized Logistic Distribution, the MGF is

$$M(t) = \frac{\Gamma(1+t)\Gamma(\alpha+t)}{\Gamma(\alpha)} \ . \qquad (2)$$

The Type II Generalized Logistic Distributions have several interesting interactions with other distributions:

5. When X distributed as a Type II Generalized Logistic Distribution, and α is near to zero, then αX acts as a random variable with standard negative exponential.
6. If $T \sim$ beta $(\alpha, 1)$, then $S = \log\{T/(1-T)\}$ distributed Generalized Logistic Distribution Type II with parameter α as a shape parameter.
7. Type II Generalized Logistic Distribution and $-\log V$ follow similar distribution, If V distributed as F with 2α and two degrees of freedom.
8. When X distributed as Type II distribution, then $X - \log(\alpha)$ and $-\log V$ has the same distribution.
9. $V \sim$ Gamma (1, 1) as α increases to infinity.
10. If Y has a standard Type II Generalized Logistic Distribution with cdf F, then F must satisfy the homogeneous differential equation below:

$$(1 - e^{-y})F' - \alpha e^{-\alpha y}(1 - F) = 0 \ . \qquad (3)$$

Using the assumptions from Balakrishnan and Hossain (2007), let X be a random variable having the below probability distribution:

$$f\,(X \mid \lambda, \alpha, \mu, \sigma) = \frac{\lambda^{\alpha}}{\sigma\Gamma(\alpha)}\ \exp[-\alpha\frac{x-\mu}{\sigma}\ ]\exp[-\lambda\frac{x-\mu}{\sigma}\ ]\ ; \tag{4}$$

$-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$, $\alpha > 0$, $\lambda > 0$, $\Gamma(.)$ gamma distribution.

If Y follows gamma distribution, then $X = \sigma \ln Y + \mu$ is double exponential.

Let λ has gamma distribution:

$$g(\lambda) = e^{-\lambda}\ \lambda > 0,$$

$$f\,(x \mid \lambda, \alpha, \mu, \sigma) = \int_0^{\infty} f(x|\lambda)g(\lambda)d\lambda$$

$$= \frac{e^{-\alpha(\frac{x-\mu}{\sigma})}}{\sigma\Gamma(\alpha)}\ \int_0^{\infty}\ \lambda^{\alpha+1-1}e^{-\lambda(1+e^{-\left(\frac{x-\mu}{\sigma}\right)})}d\lambda\ , \tag{5}$$

$$f\,(x \mid \alpha, \mu, \sigma) \qquad = \frac{\alpha}{\sigma}\frac{e^{-\alpha(\frac{x-\mu}{\sigma})}}{(1+e^{-\left(\frac{x-\mu}{\sigma}\right)})^{\alpha+1}}\quad ;\quad -\infty < x < \infty. \tag{6}$$

where the previous equation (7) represents the probability function of Type II Generalized Logistic Distribution.

If $\alpha = 1$, then equation (7) will present the standard logistic density function.

To find the CDF of Type II Generalized Logistic Distribution:

$$F(x \mid \alpha, \mu, \sigma) = \frac{\alpha}{\sigma}\int_{-\infty}^{x}\frac{1}{\sigma}\frac{e^{-\alpha(\frac{u-\mu}{\sigma})}}{(1+e^{-\alpha\left(\frac{u-\mu}{\sigma}\right)})^{\alpha+1}}\ du. \tag{7}$$

If we let $t = 1/1 + e^{-\left(\frac{u-\mu}{\sigma}\right)}$, we got:

$$F(x \mid \alpha, \mu, \sigma) = 1 - \left(\frac{e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)}\right)^{\alpha}\ ;\ -\infty < x < \infty. \tag{8}$$

Let

$$Z = \frac{x-\mu}{\sigma}, \tag{9}$$

then Z has standard Type Generalized Logistic Distribution given by

$$f(z|\alpha) = \frac{\alpha e^{-(z\alpha)}}{\left(1+ e^{-(z)}\right)^{\alpha+1}}; \qquad -\infty < z < \infty, \quad (10)$$

$$F(z|\alpha) = 1 - \left(\frac{e^{-(z)}}{\left(1+ e^{-(z)}\right)}\right)^{\alpha}; \qquad -\infty < z < \infty. \quad (11)$$

The difficulty of evaluating unknown parameters in lifetime distributions, as well as the accompanying survival and hazard functions, using a progressive censored sample has lately gained a lot of attention. The scheme of progressive censoring is of important value in life-testing experiments. It allows the elimination of live units from the experiment at diverse levels. This will potentially save plenty for the experimenter in terms of cost and time. So, the Generalized Logistic Distribution Type II with one shape parameter under Progressive Type II right censoring is approached lately to broaden the scope of distribution in situations in which the data is asymmetric.

Maximum likelihood was utilized in certain research to estimate the unknown parameters of such distributions. In this study we want to make inference about the Logistic Distribution, in specific generalized with classification of Type II Generalized Logistic Distribution under special censoring technique, Type II progressive. Applying Bayesian and classical methods. Then, to compare the results with the previous studies.

Figure 3 illustrates the distribution of the Generalized Logistic Distribution which can be skewed to the left, if α parameter is less than1or the right, if α parameter is greater than 1. Or symmetric α = 1 which is the logistic distribution. In the figure we have the blue plot when $\mu = 0, \sigma = 1$ *and* $\alpha = 0.5 < 1$. While the red plot when $\mu = 0, \sigma = 1$ *and* $\alpha = 2 > 1$. Finally, the black plot when $\mu = 0, \sigma = 1$ *and* $\alpha = 1$.

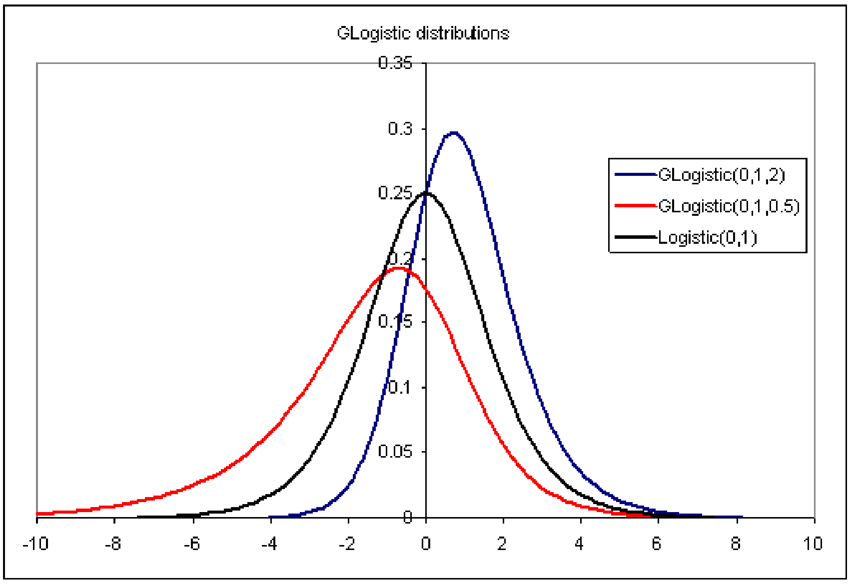

Figure 3: Generalized Logistic Distribution with Different Values of α

<https://www.vosesoftware.com/riskwiki/Generalisedlogisticdistribution.php>

1.3. Literature Review

1.3.1. Overview

The goal of a literature review is to get a better grasp of current research on a particular subject or issue. Many scholars have studied the inference of unknown parameters of skewed distributions throughout the years. The main goal of these conclusions is to find the best estimators for the unknown Generalized Logistic Distribution parameters.

1.3.2. Studies based on Generalized Logistic Distribution

The logistic distribution has been utilized in growth models as well as in a sort of regression called logistic regression. It can also be used to model actual life data. This distribution is often used as a parametric model in survival analysis for occurrence whose rate increases early and then drops subsequently, such as cancer mortality after diagnosis or treatment. It's also been used to represent stream flow and precipitation in hydrology, as well as a simplified model of wealth or income distribution in economics. In addition, the Generalized Logistic family of distributions is a particular instance of the proportional reversed hazard rate model. Types I, II, III, and IV of generalizations of the Logistic Distribution have been presented in the literature and explored by Balakrishnan and Leung (1988), Balakrishnan (1992), and Johnson et al. (1995).

Due to the importance of generalized distributions especially the Generalized Logistic Distribution, several authors studied it under complete sample or censored data. they tried to estimate it's unknow parameters following several methods. For example, Balakrishnan (1990) considered estimation of the unknown parameters of the Type I Generalized Logistic Distribution. Since it’s not easy to find the MLE explicitly for the location and scale parameters under a complete or Type II censored samples, the author applied a suitable approximation method to provide a straightforward approach for obtaining explicit estimators. To evaluate the covariances and variances of these estimators, the author obtained approximate expressions. These estimators are also shown to be as efficient as the best linear unbiased estimators (BLUE's). Also, Alkasasbeh and Raqab (2009) extensively studied the Generalized Logistic Distribution under complete samples. The goal of their work is to look at several sorts of estimating processes and see how estimators for various unknown parameters/parameters behave when using various sample sizes and parameter values. By running large numerical simulations, they compared the maximum likelihood estimators, the technique of moments estimators, estimators based on percentiles, least squares estimators, weighted least squares estimators, and L-moments estimators. As a result of the evaluation of the performance of each estimator based on bias, the researchers found out that in all of the scenarios the least squares estimators (LSE) were the best and most efficient estimators. The weighted least squares estimators WLSE's were pretty similar in terms of performance. with small sample sizes, in practically all scenarios evaluated for estimating both parameters, it is evident that percentile estimators (PCE's) work better. They conclude that, when dealing with small sample sizes it’s better to use the PCE and the LSE for medium and large sample sizes. Moreover, the Generalized Logistic Distribution was studied based on Left Type-II Censoring by Sindhu, Aslam and Hussain (2016). They considered the Bayes inference and corresponding risks to estimate the unknown parameters of the Generalized Logistic Distribution applying several asymmetric loss functions, under the assumption of different informative and non-informative prior distributions. It is also described how to extract hyperparameters using a prior prediction method. They also obtain the credible Intervals and the formulation for posterior predictive distributions. As an experiment, these estimators are compared using a simulation process as well as real data examples with graphical findings. The study's findings suggest that Bayes estimation using the gamma prior is preferable.

1.3.3. Studies based on Progressive Censoring Sampling

Considerable attention has been paid in the literature to inference in parametric distributions under a progressive censored data. Balakrishnan and Sandhu (1996) considered Type II progressive censored sample to find the best linear unbiased to estimate the parameters of the exponential distributions. In addition, they found the maximum likelihood estimators (MLE’s) and found that they are equal to the BLUE’s of the exponential distribution with two parameters.

The generalized exponential distribution was studied by Kundu and Pradhan (2009). They looked at Bayesian inference to estimate the parameters sampled from progressively censored data using distinct gamma priors for the parameters. The Lindley's Approach and Importance Sampling techniques are employed to approximate the Bayes estimates using MCMC (Monte Carlo Markov Chain). The authors noted that the Bayes estimates have strong advantages over the MLEs, if suitable prior information is available. The generalized Rayleigh distribution was considered by Maiti and Kayal (2019) where they looked at parameter estimation and reliability features in a progressive censoring- Type II sample. The MLEs and Bayes inference of the unknown parameters were evaluated under various loss functions. Salah (2020) considered evaluating unknown parameters of α-power exponential distribution based on Type II progressively censored data using the MLEs. Researcher found the approximate best linear unbiased estimators (ABLUE's) as an initial guess of the MLEs. The author discovered that ABLUEs and MLEs are so closely related of the exponential distribution with two parameters. This closeness provides good initial estimates of MLEs.

1.3.4. Studies based on Generalized Logistic Distribution Under Progressive Censoring Sampling

The Type I Generalized Logistic Distribution was considered by Asgharzadeh (2006) where he obtained an approximation to the maximum likelihood estimator and the percentage points of some pivotal that are used to construct intervals for the parameters. The Type II Generalized Logistic Distribution was considered by Balakrishnan and Hossain (2007) considered under progressive Type II censoring, with single shape parameter. The authors stated several characteristics of this distribution as well as its relationship with other distributions which will help in the modelling process. They used point estimation for the location and scale parameters using progressive Type II censoring. The (ML) and an approximate ML technique are derived and investigated. In a simulation study, multiple progressive censor scheme and sample sizes were used. They found that the approximate maximum likelihood estimators and the MLEs have similar performance in terms of bias and variance. The IV Generalized Logistic Distribution was studied by Nassar and Elmasri (2012). They considered, among other things, parameter estimation using the method of maximum likelihood and the method of moments. Aly and Bleed (2013) considered Bayesian estimation of the Generalized Logistic Distribution based on progressively censored data under accelerated life testing. Azizpour and Asgharzadeh (2018) considered Type-II hybrid progressively censored data when the lifetime distributions of the items follow Type-II Generalized Logistic Distribution. They derived the Maximum likelihood estimators and investigated their performance in estimating the location and scale parameters. They provide approximate maximum likelihood estimators and proposed asymptotic confidence intervals n addition to bootstrap confidence interval are proposed. A new Generalized Logistic Distribution was proposed by Aljarrah et al. (2020) that include several distributions as special cases. They obtained the maximum likelihood estimators of the parameters and investigated the small sample performance of the estimators. It looks that little thought has gone into analyzing the performance of Bayesian and linear estimators of the Type II Generalized Logistic Distribution parameters based on progressive censoring. In this work, we derived and investigated the performance of these estimators using simulation techniques. Moreover, we applied the results to real data sets to illustrate the applications of the methods developed in this thesis.

1.4. Research Problem Statement

A generalized model is more versatile than a conventional model in general, and many data analysts prefer it when evaluating statistical data. The Generalized Logistic Distribution has been used to model extreme variables, such as share return fluctuations (Gettinby et al., 2004) and sea level variations (van Gelder et al., 2000). It has been widely used for maximum rainfall modeling, and it is the standard model for flood frequency estimation in hydrological risk assessments in the UK and internationally (Centre for Ecology & Hydrology, 1999).

One of the most common censoring strategies used in clinical investigations, reliability trials, product quality control, and industrial tests is Type-II progressive censoring. Several researchers have recently expressed an interest in investigating parameter inference for various distributions using a progressive censoring Type-II scheme (PC) (Kundu (2008), Pradhan and Kundu (2008), Maurya et al. (2019), and Bdair et al. (2020)). In this technique, the number of observed failures $x_1, x_2, \dots, x_m$, number of complete observations to be observed m, and the censoring removal scheme, $(R_1, R_2, \dots, R_m)$, are given in this procedure.

Azizpour and Asgharzadeh (2018), for example, did some relevant work based on Generalized Logistic Distribution. Using the MLE approach, the parameters of the Generalized Logistic Distribution were estimated using Type-II hybrid progressively censored data. Furthermore, Alkasasbeh and Raqab (2009) used five estimate approaches to estimate the unknown parameters of a Generalized Logistic Distribution in the case of a full sample, including inference, applying maximum likelihood technique, to estimate the unknown parameters of a Generalized Logistic Distribution.

There is a limited amount of job that can be found based on a censored scheme from a Type II progressive sample from a distinct lifespan distribution; for instance, Balakrishnan and Hossain (2007) completed the most important study in this subject on censored data. The MLE technique and estimated MLE methods were used to estimate the unknown parameters of a Generalized Logistic Distribution based on this Type II of censoring scheme. Fernandez (2004) has covered the MLE and Bayesian approaches for estimating the parameters of the exponential distribution in general Type II progressive censoring data in his study. Furthermore, Balakrishnan and Sandhu (1996) used the MLE and BLUE approaches to investigate the exponential distributions model for estimating parameters based on a generic progressive Type II censored sample. Kundu and Pradhan took into account the most recent research (2009). Furthermore, Balakrishnan and Sandhu (1996) used the MLE and BLUE approaches to investigate the exponential distributions model for estimating parameters based on a generic progressive Type II censored sample. Kundu and Pradhan took into account the most recent research (2009) to inference the generalized exponential distribution parameters by applying the Lindley's approaching and Importance Sampling technique under the progressively censored data. Although quite a bit of work has been done on comparing different estimators to infer parameters of distributions under -Type II progressive censoring scheme in the available studies. But there is no research comparing different estimators for general logistic distribution using Type II progressive censoring.

As a result of this task constraint, this study will consider ML, Bayesian, BLUE and BLEE estimation techniques to estimate parameters under Type II progressive censored data and conclude the most efficient estimator.

1.5. Research Objective and Significant

This study is an extension of the study made by Balakrishnan and Hossain (2007) and Kundu and Pradhan (2009). In Balakrishnan and Hossain (2007), the researchers focused on the logistic distribution that was generalized with classification of Type II Generalized Logistic Distribution using Type II as a class of progressive censoring, and Kundu and Pradhan (2009) studied Bayesian estimates of the parameters under progressively censored with generalized exponential distribution.

This study considers Maximum Likelihood Estimator, Bayesian inferences and Linear estimations under progressive Type II censored data to estimate unknown parameters (location as μ and scale as σ) of Type II Generalized Logistic Distribution.

Since it is difficult to calculate the double integration when applying the Bayesian method analytically, we applied Lindley's approach and Importance Sampling techniques. We assume informative prior distributions as $\pi_1(\mu) \propto 1$ and $\pi_2(\sigma/\mu) \propto 1/\sigma$ and try to find posterior distribution using the likelihood function from Balakrishnan and Hossain (2007). Using this posterior distribution to find point estimators for the unknown parameters.

We considered linear estimation, Best Linear Unbiased Estimator (BLUE) and Best (Affine) Linear Equivariant Estimator (BLEE), to evaluate the unknown parameters of logistic distribution that is generalized with Type II category. Then we compared the results of linear estimation with the Bayesian and ML estimators.

The results of the best estimator with lowest biased and mean squared error (MSE) will be of great benefit to estimate unknown parameters (location as and scale as) of Type II Generalized Logistic Distribution. This will broaden the scope of the distribution in situations in which the data is asymmetric. Also, it will demonstrate the adaptability and importance of the Type II Generalized Logistic Distribution in survival analysis.

1.6. Research Specific Objectives

The following particular objectives will be investigated in this study:

1. Obtaining the Maximum Likelihood Estimators of unknown parameters (location and scale) of Generalized Logistic Distribution of Type II.
2. Obtaining the Bayesian (Lindley's approach and Importance Sampling) point estimators of unknown parameters (location and scale) of Generalized Logistic

Distribution of Type II.

3. Obtaining the Linear inference (BLUE and BLEE) of the parameters (location and scale) of the Generalized Logistic Distribution of Type II.

4. Comparing the Bayesian and Linear estimators with the MLEs using MSE and bias.

5. Applying the inference techniques developed in this thesis for the Generalized Logistic Distribution to real or simulated life data sets.

1.7. Scope of Study

As previously stated, the work in this thesis focused on estimating the unknown parameters of the Generalized Logistic model using classic inference, Bayesian and Best Linear Unbiased Estimator techniques in a Type II progressive censored sample. Chapter 1 introduces several basic topics in the subject of survival analysis, such as survival analysis, censoring types and schemes, and the basis of the Generalized Logistic Parametric Model. Chapter 1 also includes assessments of the literature on estimating methods such as MLE, Lindley's approach, Importance Sampling, and BLUE, as well as Type II progressive censoring. In Chapter 2, the maximum likelihood estimator, Bayes inference, BLUE and BLEE will be studied for the Generalized Logistic model under Type II progressive censored sample. In addition, in Chapter 2, a comparison of estimators based on bias and MSE will be presented. In Chapter 3, we executed a Monte Carlo simulation to discuss and analyzed the results. In Chapter 4, we explored a real-data applications of Type II progressive censorship. Finally, Chapter 5 provided a summary, conclusion, and recommendations for further research.

CHAPTER2: MAXIMUM LIKELIHOOD, BAYES ESTIMATION AND BEST UNBIASED ESTIMATOR

2.1. Introduction

We showed a general overview about maximum likelihood approach, likelihood based on progressive Type II censored data followed by the maximum likelihood method. Then we introduced the Bayesian approach and the Bayesian method. After that we explained the lindley’s approximation and the importance sampling techniques. Also, we introduced the linear approach under progressive censoring. As well as an illustration of the best linear unbiased estimator method and best (affine) linear equivariant estimator method.

2.2. The Maximum Likelihood Approach

Maximum likelihood estimation is a widely used statistical inference technique for estimating parameters in probabilistic data generation models. This approach, which is theoretically simple, yields parameter estimates with strong statistical features. The MLEs for the two unknowns μ *and* σ parameters exhibit highly intriguing asymptotic features, as determined by Lehmann et al. (1998). Using this approach, the estimators are consistent, best asymptotically normal, and asymptotically unbiased. Even when a single consistent root of the likelihood equation is known to exist, it is sometimes hard to construct a clear solution for the maximum likelihood estimate of a parameter as a function of the sample. In such situations, numerical methods must be used to assess the MLE by repeated iteration. The root of an equation can be found using a lot of numerical methods. One of the most extensively used methods for root detection is the Newton-Raphson approach. Newton's approach may be easily expanded to the challenge of finding solutions to a system of non-linear equations. Furthermore, as we get closer to the root, the strategy may be proven to be quadratically convergent.

2.2.1. Likelihood Based on Progressive Type II Censored Data

Assume we have a random variable X with a two-parameter GL distribution. So, it has the probability density function as Equation 6 and the cumulative distribution function as Equation 8:

Assume n identical objects are presented on a test, each with the Generalized Logistic (μ, σ) lifespan distribution. We have the observations sample below with a progressive censoring scheme ($R_{1,}R_2$, ..., R_m) Type II: $X_{1:m:n}$, $X_{2:m:n}$, ... , $X_{m:m:n}$. Then the joint PDF of $X_{1:m:n}$, $X_{2:m:n}$, ... , $X_{m:m:n}$ is

$$f_{X_{1:m:n},X_{2:m:n},\dots,X_{m:m:n},}(x_{1,}x_2,\dots,x_m) = \text{C} \prod_{i=1}^{m} \text{f(xi)}[1 - \text{F(xi)}]^{R_i}, \tag{12}$$

where $0 < x_{1<}x_2 < \dots < x_m < \infty$,

where the PDF and CDF of this sample can be denotes as f(.) and F(.) , respectively, as given in Equations (6) and (8) and

$$\text{C} = \text{n}(\text{n} - R_1 - 1)\cdots(\text{n} - R_1 - R_2 - \dots - R_{m-1} - \text{m} + 1). \tag{13}$$

For more details, see Balakrishnan and Aggarwala (2000) . We assume that $R_{1,}R_2$, ... , R_m the scheme, n the sample size, m the failure numbers, are fixed in advance, and

$$\text{D} = (X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}). \tag{14}$$

2.2.2. Maximum Likelihood Method

Relay on the items D as mentioned previously (17), then the likelihood functions of μ and σ can be calculated as:

$$L = k * \{\prod_{i=1}^{m} f(z_{i:m:n})[1 - F(z_{i:m:n})]^{R_i}\}, \tag{15}$$

where $z_{i:m:n} = (X_{i:m:n} - \mu)/\sigma$.

We substitute with equation (7) and (9):

$$L = k * \prod_{i=1}^{m}\left\{\left(\frac{\alpha}{\sigma}\right) * \left(\frac{e^{-\alpha\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha+1}}\right)\left[\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\right]^{R_i}\right\}$$

$$= k * \frac{\alpha^m}{\sigma^m}\prod_{i=1}^{m}\left\{\left(\frac{e^{-\alpha\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha+1}}\right)\left[\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\right]^{R_i}\right\}. \quad (16)$$

Substitute with z = $\frac{x-\mu}{\sigma}$ to get:

$$L = k * \frac{\alpha^m}{\sigma^m}\left\{\prod_{i=1}^{m}\frac{e^{-(-z_i\alpha)}}{\left(1+e^{-(z_i)}\right)^{\alpha+1}}\prod_{i=1}^{m}(1-(1-\left\{\frac{e^{-(z_i)}}{1+e^{-(z_i)}}\right\}^{\alpha}))^{R_i}\right\}$$

$$= k * \frac{\alpha^m}{\sigma^m}\left\{\prod_{i=1}^{m}\frac{e^{-(-z_i\alpha)}}{\left(1+e^{-(z_i)}\right)^{\alpha+1}}\prod_{i=1}^{m}\left\{\left\{\frac{e^{-(z_i)}}{1+e^{-(z_i)}}\right\}^{\alpha}\right\}^{R_i}\right\}.$$

Rearranged the equation, we had the following:

$$L = k * \frac{\alpha^m}{\sigma^m}\left\{\prod_{i=1}^{m}\left\{\left\{\frac{e^{-(z_i)}}{1+e^{-(z_i)}}\right\}^{\alpha}\right\}^{R_i+1}\prod_{i=1}^{m}\frac{1}{1+\exp(-z_i)}\right\}. \quad (17)$$

The ln of the likelihood function is:

$$l = ln\left\{k * \left\{\prod_{i=1}^{m} f(z_{i:m:n})\{1-F(z_{i:m:n})\}^{R_i}\right\}\right\}$$

$$= ln\left\{k * \frac{\alpha^m}{\sigma^m}\left\{\prod_{i=1}^{m} f(z_{i:m:n})\{1-F(z_{i:m:n})\}^{R_i}\right\}\right\}. \quad (18)$$

Applying the ln for the likelihood function:

$$l = const. + mln\alpha - mln\sigma * \sum_{i=1}^{m} lnf(z_i) + \sum_{i=1}^{m} R_i \ln(1 - F(z_i)),$$

$$= lnk + mln\alpha - mln\sigma + \left\{\sum_{i=1}^{m}(R_i+1)ln\left\{\left\{\frac{e^{-(z_i)}}{1+e^{-(z_i)}}\right\}^{\alpha}\right\}\right\} + \left\{\sum_{i=1}^{m} ln\left\{\frac{1}{1+e^{-(z_i)}}\right\}\right\}. \quad (19)$$

To find the maximum likelihood estimator, we need to find the first and second derivatives. To simplify the equation. Let:

$$\Delta_1(z) = \frac{\partial}{\partial z}\ln f(z) = \left\{\frac{e^{-z} - \alpha}{1+e^{-z}}\right\}, \quad (20)$$

$$\Delta_2(z) = \frac{f(z)}{1-F(z)} = \left\{\frac{\frac{\alpha e^{(-z\alpha)}}{\sigma\{(1+e^{-z})\}^{\alpha+1}}}{1-\left(1-\left[\frac{e^{-(z)}}{1+e^{-(z)}}\right]^{\alpha}\right)}\right\} = \frac{\alpha}{\sigma}\left\{\frac{e^{-(z)}}{1+e^{-(z)}}\right\}, \quad (21)$$

(hazard function of Type II Generalized Logistic Distribution).

$$z = (x - \mu)/\sigma \;\rightarrow \frac{\partial z}{\partial \mu} = -\frac{1}{\sigma} \;\rightarrow \frac{\partial z}{\partial \sigma} = -\frac{x-\mu}{\sigma^2} = -\frac{1}{\sigma}\, z\,. \qquad (22)$$

To find the MLE of μ and σ , we should find the first derivative equations and equalize it to 0:

$$\frac{\partial lnL}{\partial \mu} = -\frac{1}{\sigma}\sum_{i=1}^{m} \Delta_1(z_i) + \frac{1}{\sigma}\sum_{i=1}^{m} R_i\Delta_2(z_i) = 0\,, \qquad (23)$$

$$\frac{\partial lnL}{\partial \sigma} = -\frac{m}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{m} z_i\Delta_1(z_i) + \frac{1}{\sigma}\sum_{i=1}^{m} R_i z_i\Delta_2(z_i) = 0. \qquad (24)$$

Also, we need to find second derivative equations to make sure we have the minimum value where we can have the maximum likelihood estimators of μ and σ:

$$\frac{\partial^2 l}{\partial \mu^2} = \frac{1}{\sigma^2}\sum_{i=1}^{m} \Delta'_1(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i\Delta'_2(z_i), \qquad (25)$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{m}{\sigma^2} + \frac{2}{\sigma^2}\sum_{i=1}^{m} z_i\Delta_1(z_i) + \frac{1}{\sigma^2}\sum_{i=1}^{m} z_i{}^2\Delta'_1(z_i) - \frac{2}{\sigma^2}\sum_{i=1}^{m} R_i z_i\Delta_2(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i z_i{}^2\Delta'_2(z_i)\,, \qquad (26)$$

$$\frac{\partial^2 l}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2}\sum_{i=1}^{m} \Delta_1(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i\Delta_2(z_i) + \frac{1}{\sigma^2}\sum_{i=1}^{m} z_i\Delta'_1(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i z_i{}^2\Delta'_2(z_i)\,, \qquad (27)$$

where

$$\Delta'_1(z_i) = \frac{(\alpha+1)e^{-z}}{(1+e^{-z})^2}\,,$$

$$\Delta'_2(z_i) = \frac{f(z)\big(1-F(z)\big)\Delta_1(z) + f(z)}{(1-F(z))^2}\,.$$

The first and second derivative equations for μ, σ cannot be solved explicitly. So, we solved them numerically. Some iterative methods must be employed, one of these methods is Newton–Raphson approach to compute the MLEs of location and scale, a starting value around the global maximum is required.

2.3.The Bayesian Approach

The goal of Bayesian probability is to determine the probability density function along a set of hypotheses, or the parameters of a probabilistic model, given a set of data. The posterior distribution refers to this quantity of interest. A probability distribution over all possible values of a probabilistic model's parameters is known as the posterior distribution, which is one of the characteristics of the Bayesian method. In contrast, approaches like maximum likelihood estimation provide a single 'optimal' set of parameter values, known as a point estimate.

The posterior distribution is equal to the product of a constant and the likelihood, which incorporates the data's information, as well as the prior knowledge. Bayes' theorem is used in Bayes estimates:

$$\mathrm{p(param|dt)} = \frac{\mathrm{p(dt|param)p(param)}}{\mathrm{p(dt)}}; \qquad (28)$$

where:

- param : unknown parameters of a model or hypothesis.
- dt : whole data.
- p(param|dt) : joint posterior distribution.
- p(dt| param) : likelihood function.
- p(param) : prior distribution.
- p(dt) : marginal probability of the data or the evidence, where

$$p(dt) = \int p(dt, param)p(\mathrm{param})p(\mathrm{dt})\, dparamddt. \qquad (29)$$

p(dt) is a normalizing constant that depends solely on the data and doesn't have to be calculated directly in most scenarios. As a result, Bayes' theorem is frequently used in practice in the form:

$$\mathrm{p(param|dt)} \propto \mathrm{p(\,dt\mid param\,)p(param)}. \qquad (30)$$

$$\text{Posterior} \propto \text{likelihood} \times \text{prior}.$$

The Bayesian method, in its most basic version, in light of recent facts, revises the prior belief linked with a hypothesis. The prior distribution, of course, contains this prior knowledge, whereas the probability incorporates the data's effect. In both

Bayesian and frequentist statistics, Bayes' theorem (Equation 39) is a perfectly acceptable and rigorous theorem, but its conceptual meaning is unique to the Bayesian understanding of probability.

2.3.1. The Bayesian Method

Bayesian statistical methods begin with established 'prior' beliefs and update them with data to generate 'posterior' beliefs that can be used to make inferences. Based on this technique, we are going to estimate Type II Generalized Logistic Distribution location and scale parameters (μ and σ).

Assuming non informative prior distributions for Type II Generalized Logistic Distribution parameters, and applying the joint prior of the location and scale parameters,μ and σ,respectively, we got the joint distribution of data, μ and σ as:

The unknown parameters have non-informative as shown below:

$$\pi_1(\mu) = 1; \tag{31}$$

$$\pi_2(\sigma) = 1/\sigma; \tag{32}$$

The joint prior distribution of μ and σ:

$$\pi(\mu,\sigma) = \pi_1(\mu) * \pi_2(\sigma) . \tag{33}$$

As we know previously (18):

$$Likelihood\ function: \ L(data|\alpha,\mu,\sigma) \quad \alpha \quad \mathrm{k} * \prod_{i=1}^{m} f(z_{i:m:n})[1 - F(z_{i:m:n})]^{R_i}$$

To derive the joint distribution of data, μ and σ , we multiply likelihood function with the distribution of joint prior of μ and σ as follows:

$$L(data|\alpha,\mu,\sigma) * \pi(\mu,\sigma). \tag{34}$$

Based on the previous equation, the joint posterior density of, μ and σ given the data is:

$$\pi(\mu,\sigma/data) = \frac{L(data/\alpha,\mu,\sigma) * \pi(\mu,\sigma)}{\int_{\sigma=0}^{\infty}\int_{\mu=0}^{\infty} L(data|\mu,\sigma) * \pi(\mu,\sigma) d\mu d\sigma} . \tag{35}$$

In order to find the Bayes estimator of any function of the parameters μ and σ , assume $t\ (\mu\ ,\sigma)$ under the squared error loss function (SELF), we need to find:

$$\hat{t}(\mu\ ,\sigma) \quad = \quad E_{\mu,\frac{\sigma}{data}}\big(\mathrm{t}(\mu\ ,\sigma)\big);$$

$$\hat{t}(\mu\ ,\sigma) \quad = \frac{\int_{\sigma=0}^{\infty}\int_{\mu=0}^{\infty} t(\mu\ ,\sigma)*L(data|\mu,\sigma)*\ \pi(\mu,\sigma)d\mu d\sigma}{\int_{\sigma=0}^{\infty}\int_{\mu=0}^{\infty} L(data|\mu,\sigma)*\ \pi(\mu,\sigma)d\mu d\sigma}. \tag{36}$$

It is difficult to calculate the equation (47) analytically. We can approximate this Bayes estimators using the Lindley's approximation method and Importance sampling procedures.

2.3.1.1. Lindley's Approximation

In order to derive the ratio of two integrals, Lindley suggested his procedure. This approximation method deals with the integral ratio as a whole and yields a single numerical solution. Several researchers have used this approximation to obtain Bayes estimators for certain lifetime distributions.

In the following explanation, suppose we have two parameters case, using the notation (η_1, η_2) for $(\mu\ ,\sigma)$. The Lindley's approximation can be explained as:

$$\hat{t} = t(\widehat{\eta_1}, \widehat{\eta_2}) + \frac{1}{2}\ (A + \ \mathrm{l}_{30}\mathrm{B}_{12} + \ \mathrm{l}_{03}\mathrm{B}_{21} + \ \mathrm{l}_{21}\mathrm{C}_{12} + \ \mathrm{l}_{12}\mathrm{C}_{21}) + \ p_1\,\mathrm{A}_{12} + \ p_2\,\mathrm{A}_{21}, \tag{37}$$

$$A = \sum_{i=1}^{2}\sum_{j=1}^{2} w_{ij}\tau_{ij},\ \mathrm{l}_{ij} = \frac{\partial^{i+j}L(\eta_1,\eta_2)}{\partial\eta^{i}{}_{1}\partial\eta^{j}{}_{2}}\ ,\ i,j = 0,1,2,3\ \ and\ i+j = 3\ \ , \tag{38}$$

$\mathbf{p_i} = \frac{\partial \mathbf{p}}{\partial \eta_i}$, (39) $\quad \mathbf{w_i} = \frac{\partial \mathbf{t}}{\partial \eta_i}$, (40) $\quad \mathbf{w_{ij}} = \frac{\partial^2 \mathbf{t}}{\partial \eta_i \partial \eta_j}$, (41)

$\mathrm{p} = \log \boldsymbol{\pi}(\boldsymbol{\eta_1}, \boldsymbol{\eta_2})$(42) $\quad , \mathrm{A_{ij}} = \mathrm{w_i}\tau_{ii} + \ \mathrm{w_j}\tau_{ji}$, (43) $\quad \mathrm{B_{ij}} = (\mathrm{w_i}\tau_{ii} + \ \mathrm{w_j}\tau_{ji})\tau_{ii}$, (44)

$$\mathbf{c_{ij}} = \boldsymbol{\tau_{ii}\tau_{ij}} + \ \mathbf{w_j}\big(\boldsymbol{\tau_{ii}\tau_{jj}} + \mathbf{2}\boldsymbol{\tau_{ij}}^{\mathbf{2}}\big)\,. \tag{45}$$

In our case log likelihood function,

$$l \ = const.\ -m ln\,\sigma + \ \sum_{i=1}^{m} log f(z_i) + \sum_{i=1}^{m} R_i \log(1 - \ F(z_i))\ , \tag{46}$$

$$U = -\ \frac{\partial^2 L}{\partial \mu^{\,2}}\ = \frac{1}{\sigma^2}\sum_{i=1}^{m} \Delta'_1(z_i) - \ \frac{1}{\sigma^2}\sum_{i=1}^{m} R_i \Delta'_2(z_i)\ \ , \tag{47}$$

$$V = -\frac{\partial^2 L}{\partial\mu\partial\sigma} = \frac{1}{\sigma^2}\sum_{i=1}^{m}\Delta_1(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m}R_i\Delta_2(z_i) + \frac{1}{\sigma^2}\sum_{i=1}^{m}z_i\Delta'_1(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m}R_i {z_i}^2\Delta'_2(z_i)\ , \tag{48}$$

$$W = -\frac{\partial^2 L}{\partial\sigma^2} = -[\frac{m}{\sigma^2} + \frac{2}{\sigma^2}\sum_{i=1}^{m}z_i\Delta_1(z_i) + \frac{1}{\sigma^2}\sum_{i=1}^{m}{z_i}^2\Delta'_1(z_i) - \frac{2}{\sigma^2}\sum_{i=1}^{m}R_i z_i\Delta_2(z_i) - \frac{1}{\sigma^2}\sum_{i=1}^{m}R_i{z_i}^2\Delta'_2(z_i)], \tag{49}$$

$$\tau_{11} = \frac{W}{UW-V^2} = \left\{\frac{(-\frac{\partial^2 L}{\partial\sigma^2})}{\left(\left(-\frac{\partial^2 L}{\partial\mu^2}\right)*\left(-\frac{\partial^2 L}{\partial\sigma^2}\right)-(-\frac{\partial^2 L}{\partial\mu\partial\sigma})^2\right)}\right\}, \tag{50}$$

$$\tau_{12} = -\frac{V}{UW-V^2} = \left\{-\frac{(-\frac{\partial^2 L}{\partial\mu\partial\sigma})}{\left(-\frac{\partial^2 L}{\partial\mu^2}\right)*\left(-\frac{\partial^2 L}{\partial\sigma^2}\right)-(-\frac{\partial^2 L}{\partial\mu\partial\sigma})^2}\right\}\ , \tag{51}$$

$$\tau_{22} = \frac{U}{UW-V^2} = \left\{\frac{(-\frac{\partial^2 L}{\partial\mu^2})}{\left(\left(-\frac{\partial^2 L}{\partial\mu^2}\right)*\left(-\frac{\partial^2 L}{\partial\sigma^2}\right)-(-\frac{\partial^2 L}{\partial\mu\partial\sigma})^2\right)}\right\}\ , \tag{52}$$

$$l_{30} = \frac{\partial^3 L}{\partial\mu^3} = \frac{-1}{\sigma^3}\sum_{i=1}^{m}\Delta''_1(z_i) + \frac{1}{\sigma^3}\sum_{i=1}^{m}R_i\Delta''_2(z_i)\ , \tag{53}$$

$$l_{03} = \frac{\partial^3 L}{\partial\sigma^3} = \left\{\frac{-2m}{\sigma^3} - \frac{4}{\sigma^3}\sum_{i=1}^{m}z_i\Delta_1(z_i) - \frac{2}{\sigma^3}\sum_{i=1}^{m}{z_i}^2\Delta'_1(z_i) - \frac{2}{\sigma^3}\sum_{i=1}^{m}{z_i}^2{\Delta_1}'(z_i) - \frac{2}{\sigma^3}\sum_{i=1}^{m}{z_i}^2\Delta'_1(z_i) - \frac{1}{\sigma^3}\sum_{i=1}^{m}{z_i}^3{\Delta_1}''(z_i) + \frac{4}{\sigma^3}\sum_{i=1}^{m}R_i z_i\Delta_2(z_i) + \frac{2}{\sigma^3}\sum_{i=1}^{m}R_i{z_i}^2{\Delta_2}'(z_i) + \frac{2}{\sigma^3}\sum_{i=1}^{m}R_i{z_i}^2{\Delta_2}'(z_i) + \frac{1}{\sigma^3}\sum_{i=1}^{m}R_i{z_i}^3{\Delta_2}''(z_i)\right\}, \tag{54}$$

$$l_{12} = \frac{\partial^3 L}{\partial\mu\partial\sigma^2} = \left\{-\frac{2}{\sigma^3}\sum_{i=1}^{m}\Delta_1(z_i) - \frac{1}{\sigma^3}\sum_{i=1}^{m}z_i{\Delta_1}'(z_i) + \frac{2}{\sigma^3}\sum_{i=1}^{m}R\Delta_2(z_i) + \frac{1}{\sigma^3}\sum_{i=1}^{m}R_i z_i{\Delta_2}'(z_i) - \frac{2}{\sigma^3}\sum_{i=1}^{m}z_i{\Delta_1}'(z_i) - \frac{1}{\sigma^3}\sum_{i=1}^{m}{z_i}^2{\Delta_1}'(z_i) + \frac{2}{\sigma^3}\sum_{i=1}^{m}R_i z_i{\Delta_2}'(z_i) + \frac{1}{\sigma^3}\sum_{i=1}^{m}R_i{z_i}^2{\Delta_2}''(z_i)\right\}\ , \tag{55}$$

$$l_{21} = \frac{\partial^3 L}{\partial\mu^2\partial\sigma} = \left\{\frac{-2}{\sigma^3}\sum_{i=1}^{m}\Delta'_1(z_i) - \frac{1}{\sigma^3}\sum_{i=1}^{m}z_i\Delta''_1(z_i) + \frac{2}{\sigma^3}\sum_{i=1}^{m}R_i\Delta'_2(z_i) + \frac{1}{\sigma^3}\sum_{i=1}^{m}R_i z_i\Delta''_2(z_i)\right\}, \tag{56}$$

$$\Delta''_1(z_i) = \frac{(\alpha+1)e^{-z}*(1+e^{-z})^2+2(1+e^{-z})(\alpha+1)e^{-z}}{(1+e^{-z})^4}, \tag{57}$$

$$\Delta''_2(z_i) = \frac{\left\{\begin{matrix}\left[f(z)'(1-F(z))\Delta_1(z)+f(z)(1-F(z))'\Delta_1(z)+f(z)(1-F(z)){\Delta_1}'(z)+f(z)'\right]*(1-F(z))^2\\ -f(z)(1-F(z))\Delta_1(z)*(1-F(z))^{2'}\end{matrix}\right\}}{(1-F(z))^4}, \tag{58}$$

$$f(z)' = [\frac{\alpha}{\sigma}\frac{e^{-\alpha(z)}}{(1+e^{-(z)})^{\alpha+1}}]\ '$$

$$= \frac{\alpha}{\sigma} \frac{\{ (-\alpha)e^{-z\alpha} * (1+e^{-z})^{\alpha+1} + (\alpha+1)(1+e^{-z})e^{-z\alpha}\}}{((1+e^{-z})^{\alpha+1})^2}$$

$$= \frac{\alpha}{\sigma} \frac{\{ (-\alpha e^{-z\alpha})(1+e^{-z})[(1+e^{-z})^{\alpha} + e^{-z}]\}}{((1+e^{-z})^{\alpha+1})^2}, \tag{59}$$

$$\left(1-F(z)\right)' = \left[1-\left(1-\left(\frac{e^{-(z)}}{(1+e^{-(z)})}\right)^{\alpha}\right)\right]'$$

$$= \left[\left(\frac{e^{-(z)}}{(1+e^{-(z)})}\right)^{\alpha}\right]'$$

$$= \frac{(-\alpha e^{-z\alpha})}{(1+e^{-z})^{\alpha+1}}, \tag{60}$$

The joint prior distribution of μ and σ =

$$\pi(\eta_1,\eta_2) = (\pi(\mu) * \pi(\sigma)) = \left(1 * \frac{1}{\sigma}\right) = \frac{1}{\sigma}, \tag{61}$$

$$p = \ln \pi(\eta_1,\eta_2) = \ln(\pi(\mu) * \pi(\sigma)) = \ln\left(1 * \frac{1}{\sigma}\right) = \ln\left(\frac{1}{\sigma}\right) = -\ln(\sigma), \tag{62}$$

$$p_1 = \frac{\partial p}{\partial \eta_1} = \frac{\partial p}{\partial \mu} = \left\{\{\partial \ln(\pi(\eta_1,\eta_2)\}/\partial\mu\right\} = \{\partial(-\ln(\sigma))/\partial\mu\} = 0, \tag{63}$$

$$p_2 = \frac{\partial p}{\partial \eta_2} = \frac{\partial p}{\partial \sigma} = \left\{\{\partial \ln(\pi(\eta_1,\eta_2)\}/\partial\sigma\right\} = \{\partial(-\ln(\sigma))/\partial\sigma\} = (-1/\sigma). \tag{64}$$

When estimating the parameter μ**:**

$$A = 0,\ B_{12} = \tau_{11}^2,\ B_{21} = \tau_{21}\tau_{22}, \tag{65}$$

$$c_{12} = 3\tau_{11}\tau_{12},\ c_{21} = 3\tau_{11}\tau_{22} + 2\tau_{21}^2, \tag{66}$$

$$A_{12} = \tau_{11},\ A_{21} = \tau_{12}, \tag{67}$$

$$p_1 = \frac{\partial p}{\partial \boldsymbol{\mu}} = 0, \tag{68}$$

$$p_2 = \frac{\partial p}{\partial \sigma} = (-1/\sigma). \tag{69}$$

When estimating the parameter σ**:**

$$A = 0,\ B_{12} = \tau_{11}\tau_{12},\ B_{21} = \tau_{22}^2, \tag{70}$$

$$c_{12} = \tau_{11}\tau_{22} + 2\tau_{12}^2,\ c_{21} = 3\tau_{21}\tau_{22}, \tag{71}$$

$$A_{12} = \tau_{21},\ A_{21} = \tau_{22}, \tag{72}$$

$$p_1 = \frac{\partial p}{\partial \boldsymbol{\mu}} = 0, \tag{73}$$

$$p_2 = \frac{\partial p}{\partial \sigma} = (-1/\sigma). \quad (74)$$

So, the Lindley's approximation of the μ :

$$\widehat{\mu_B} = \widehat{\mu} + \frac{1}{2} * \begin{pmatrix} 0 + l_{30}{\pi_{11}}^2 + l_{03}\pi_{21}\pi_{22} + 3l_{21}\pi_{11}\pi_{12} + \\ l_{12(}\pi_{22}\pi_{11} + 2{\pi_{21}}^2) \end{pmatrix}$$

$$+(0) * \pi_{11} + \left(-\frac{1}{\sigma}\right) * \pi_{12} \quad . \quad (75)$$

And the Lindley's approximation of the σ is:

$$\widehat{\sigma_B} = \widehat{\sigma} + \frac{1}{2} * \big(0 + l_{30}\pi_{12}\pi_{11} + l_{03}{\pi_{22}}^2 +$$

$$l_{21(}\pi_{22}\pi_{11} + 2{\pi_{12}}^2) + 3l_{12}\pi_{22}\pi_{21}\big) + (0) * \pi_{21} + \left(-\frac{1}{\sigma}\right) * \pi_{22} \quad (76)$$

where $\widehat{\mu}$ and $\widehat{\sigma}$ are MLE of μ , σ respectively.

2.3.1.2.Importance Sampling

It is a common variance reduction technique. It can be explained as a weighted average of random samples taken from another distribution h_v(x) ("importance sampling" density function) to estimate an expectation with respect to the target density function f_x(x). The significance sampling function must be chosen carefully to give better estimation result.

Applying the previous assumption, the prior distribution of μ and σ is non-informative priors for the location and scale parameters (μ and σ) (Equations 42,43,44):

$$\pi_1(\mu) = 1;$$

$$\pi_2(\sigma) = \frac{1}{\sigma};$$

$$\pi(\mu, \sigma) = 1 * 1/\sigma = 1/\sigma.$$

The posterior function can be derived as the following:

$$\pi(\mu, \sigma | data) = L(data | \alpha, \mu, \sigma) * \pi(\mu, \sigma), \quad (77)$$

$$\pi(\mu, \sigma | data) = \text{k} * \prod_{i=1}^{m} f(z_{i:m:n})[1 - F(z_{i:m:n})]^{r_i} * \pi(\mu, \sigma),$$

$$\pi(\mu,\sigma|data) = k\frac{\alpha^m}{\sigma^m}\prod_{i=1}^{m}\left\{\left(\frac{e^{-\alpha\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha+1}}\right)\left[1-\left[1-\left[\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\right]\right]\right]^{R_i}\right\}\times$$

$$1\times\frac{1}{\sigma}\ ,$$

$$= k\frac{\alpha^m}{\sigma^m}\prod_{i=1}^{m}\left\{\left(\frac{e^{-\alpha\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha+1}}\right)\left[\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\right]^{R_i}\right\}\times 1\times\frac{1}{\sigma}\ ,$$

$$= k\frac{\alpha^m}{\sigma^{m+1}}\prod_{i=1}^{m}\left\{\frac{1}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)}\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\left[\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\right]^{R_i}\right\},$$

$$= k\frac{\alpha^m}{\sigma^{m+1}}\prod_{i=1}^{m}\left\{\frac{1}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)}\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha(R_i+1)}\right\},$$

$$\propto \frac{1}{\sigma^{m+1}}e^{-\frac{\sum_{i=1}^{m}(x_i-\mu)}{\sigma}}\prod_{i=1}^{m}\left\{\frac{e^{-(\alpha(R_i+1)-1)\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha(R_i+1)+1}}\right\}\ ,$$

$$\propto \frac{1}{\sigma^{m+1}}e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\prod_{i=1}^{m}\left\{\frac{e^{-(\alpha(R_i+1)-1)\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha(R_i+1)+1}}\right\},$$

$$\propto \frac{1}{\sigma^m}\frac{m}{\sigma}\frac{e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}}{\left(1+e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\right)^2}\left(1+e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\right)^2\prod_{i=1}^{m}\left\{\frac{e^{-(\alpha(R_i+1)-1)\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha(R_i+1)+1}}\right\},$$

$$\propto \left\{\frac{m^{m-1}}{\Gamma(m-1)}\left(\frac{1}{\sigma}\right)^m e^{-m/\sigma}\right\}\left\{\frac{m}{\sigma}\frac{e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}}{\left(1+e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\right)^2}\right\}\left\{\frac{e^{m/\sigma}}{m^{m-1}}\left(1+\ e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\right)^2\prod_{i=1}^{m}\left\{\frac{e^{-(\alpha(R_i+1)-1)\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha(R_i+1)+1}}\right\}\right\}.$$

(78)

Here we have:

$$f_T(\sigma) = \left\{\frac{m^{m-1}}{\Gamma(m-1)}\left(\frac{1}{\sigma}\right)^m e^{-m/\sigma}\right\} \tag{79}$$

This is the inverse gamma distribution's pdf with parameters $m-1$ and m.

$$f_T\,(\mu) = \left\{\frac{m}{\sigma}\frac{e^{\frac{\mu-\bar{x}}{\sigma/m}}}{\left(1+e^{\frac{\mu-\bar{x}}{\sigma/m}}\right)^2}\right\} \tag{80}$$

This is the logistic distribution with parameters $\bar{x} = \frac{\sum_{i=1}^{m} x_i}{m}$ and σ/m.

$$h = \left\{\frac{e^{m/\sigma}}{m^{m-1}}\left(1 + e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\right)^2 \prod_{i=1}^{m}\left\{\frac{e^{-(\alpha(R_i+1)-1)\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha(r_i+1)+1}}\right\}\right\} \tag{81}$$

We can rewrite the posterior function as:

$$\pi\left(\mu, \frac{\sigma}{data}\right) \propto\ f_T\,(\mu) *\ f_T\,(\sigma) * h(\mu,\sigma). \tag{82}$$

Using (95), we can use the importance sampling to find the Bayes estimates of any function of μ and σ ¸for example t (μ,σ):

We can start with the right-hand side of (95) and mark it as $\pi_N(\mu,\sigma|data)$. So, the difference between $\pi_N(\mu,\sigma|data)$ and $\pi(\mu,\sigma|data)$ is only the proportionality constant. The, Bayes estimate of $t(\mu,\sigma)$ under the squared error loss function is

$$\hat{t}(\mu\,,\sigma) \quad = \frac{\int_{\sigma=0}^{\infty}\int_{\mu=0}^{\infty} t(\mu\,,\sigma)*\pi_N(\mu,\sigma|data)\; d\mu d\sigma}{\int_{\sigma=0}^{\infty}\int_{\mu=0}^{\infty} \pi_N(\mu,\sigma|data)\; d\mu d\sigma}. \tag{83}$$

It is obvious from (96) that using the importance sampling procedure to approximate $\hat{t}(\mu\,,\sigma)$, this does not require to determine the normalizing constant. In order to find the estimate of the Generalized Logistic Distribution parameters we do the following steps:

Step One:

Generate σ from inverse gamma distribution with parameters $m-1$ and 1

Step Two:

Generate μ from the logistic distribution with parameters $\bar{x} = \frac{\sum_{i=1}^{m} x_i}{m}$ and σ/m, where σ is generated from Step One.

Repeat this step to obtain $((\mu_1, \sigma_1), (\mu_2, \sigma_2), \ldots, (\mu_N, \sigma_N))$

Step Three:

The approximate value of parameters ($\hat{\mu}$, $\hat{\sigma}$) can be obtained as:

$$\sum_{i=1}^{N} t(\mu_i, \sigma_i) h((\mu_i, \sigma_i) \,/\, \sum_{i=1}^{N} h((\mu_i, \sigma_i) \quad . \tag{84}$$

2.4.The Linear Approach Under Progressive Censoring

Linear statistics has an easy and accurate structure. Researchers have been very interested in using linear inference for parametric distributions with ordered data in a variety of applications as a result of this.

Suppose we have X = $(X_{1:m:n}, \ldots, X_{m:m:n})$` be a randomly generated vector of Type-II censored order statistics with location parameter $\mu$ and scale parameter $\sigma$. And let Y = $(Y_{1:m:n}, \ldots, Y_{m:m:n})$` such that:

$$Y_{j:m:n} = \frac{X_{j:m:n} - \mu}{\sigma}, \text{ where j = 1, ..., m.} \tag{85}$$

Let W = σ(Y – E(Y)), b = E(Y), $\theta = (\mu \;\; \sigma)$` and B = [$\mathbb{I}$, b]. (86)

So, X can be presented as a linear equation:

$$X = \mu . \mathbb{I} + \sigma . Y = \mu . \mathbb{I} + \sigma . E(Y) + W = [\mathbb{I}, b]\binom{\mu}{\sigma} + W = B\,\theta + W . \tag{87}$$

Assuming Σ is regular, and non-singular covariance matrix.

Σ is the covariance matrix cov(Y),

$$\Sigma = \Delta \, \Sigma_{U^{\mathcal{R}}} \, \Delta . \tag{88}$$

2.4.1. Best Linear Unbiased Estimator

Estimating local and scale parameters model, we suppose we have $m \geq 2$ and $n = \sum_{j=1}^{m} r_j + 1$, so BLUE estimator of μ :

$$\hat{\mu}_{LU} = \frac{1}{\Delta} . ((b`\Sigma^{-1} b)(\mathbb{I}` \Sigma^{-1} X) - (\mathbb{I}` \Sigma^{-1} b)(b`\Sigma^{-1} X)) , \tag{89}$$

$$\hat{\sigma}_{LU} = \frac{1}{\Delta} . ((\mathbb{I}`\Sigma^{-1} \mathbb{I})(b` \Sigma^{-1} X) - (\mathbb{I}` \Sigma^{-1} b)(\mathbb{I}`\Sigma^{-1} X)) , \tag{90}$$

$$\text{Where } \Delta = ((\mathbb{I}`\Sigma^{-1}\,\mathbb{I})(\,b`\,\Sigma^{-1}b) - (\mathbb{I}`\Sigma^{-1}\,b)^2 > 0 \quad (91)$$

In order to find the covariance matrix, we used the following algorithm:

1. Let $\gamma_j = n - j + 1, \qquad j = 1,2,\dots,n.$ (92)
2. Calculate $c_r = \prod_{j=1}^{r} \gamma_j, \qquad r = 1,2,\dots,m.$ (93)
3. Calculate $d_r = \prod_{j=1}^{r} (\gamma_j + 1), \qquad r = 1,2,\dots,m.$ (94)
4. Calculate $e_r = \prod_{j=1}^{r} (\gamma_j + 2), \qquad r = 1,2,\dots,m.$ (95)
5. Calculate $a_r = \frac{d_r}{e_r}, \qquad r = 1,2,\dots,m.$ (96)
6. Calculate $b_r = \frac{c_r}{d_r}, \qquad r = 1,2,\dots,m.$ (97)
7. Calculate $EU_r = \Pi_r = 1 - b_r, \qquad r = 1,2,\dots,m.$ (98)
8. Calculate $COVU_rU_s = (a_r - b_r)b_s, r = 1,\dots,m, s = 1,\dots,m,$ (99)

(This will give the Matrix $\Sigma_{U^{\mathcal{R}}}$).

9. Calculate the diagonal matrix Δ with diagonal elements

$$\left(\frac{1}{f(F^{-1}(\Pi_1))}, \dots, \frac{1}{f(F^{-1}(\Pi_r))}\right),$$

where from (Equations 7 & 9):

$$f(x) = \left(\frac{e^{-\alpha\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha+1}}\right),$$

and

$$F(x) = \left[1 - \left[\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\right]\right].$$

10. Calculate the required covariance matrix (Equation 101)

$$\Sigma = \Delta\, \Sigma_{U^{\mathcal{R}}}\, \Delta.$$

2.4.2. Best (affine) Linear Equivariant Estimator

Estimating local and scale parameters model:

$$\hat{\mu}_{LE} = \frac{1}{\Delta_1}.((b`\Sigma^{-1} b + 1)( \mathbb{I}` \Sigma^{-1}X) - (\mathbb{I}` \Sigma^{-1}b)(b`\Sigma^{-1} X)). \quad (100)$$

$$\hat{\sigma}_{LE} = \frac{1}{\Delta_1}.((\mathbb{I}`\Sigma^{-1} \mathbb{I})( b` \Sigma^{-1}X) - (\mathbb{I}` \Sigma^{-1}b)(\mathbb{I}`\Sigma^{-1} X)). \quad (101)$$

where $\Delta_1 = \Delta + ((\mathbb{I}`\Sigma^{-1} \mathbb{I})$. (102)

$$E(\hat{\mu}_{LE}) = \mu + \frac{(\mathbb{I}` \Sigma^{-1}b)}{\Delta_1} * \sigma. \quad (103)$$

$$E(\hat{\sigma}_{LE}) = \frac{\Delta}{\Delta_1} * \sigma . \quad (104)$$

$$MSE(\hat{\mu}_{LE}) = \frac{(b`\Sigma^{-1} b+1)}{\Delta_1} * \sigma^2 . \quad (105)$$

$$MSE(\hat{\sigma}_{LE}) = \frac{(\mathbb{I}`\Sigma^{-1} \mathbb{I})}{\Delta_1} * \sigma^2. \quad (106)$$

$$E(\hat{\mu}_{LE} - \mu) * (\hat{\sigma}_{LE} - \sigma) = \frac{(\mathbb{I}` \Sigma^{-1}b)}{\Delta_1} * \sigma^2. \quad (107)$$

The estimators, the Bayesian, and Linear estimators from the previous steps with the MLE, will be examined in terms of bias and mean squared errors defined by:

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta, \quad (108)$$

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2 \quad (109)$$

$$= Var(\hat{\theta}) + Bias^2(\hat{\theta}). \quad (110)$$

CHAPTER 3: MONTE CARLO STUDY RESULTS AND FINDINGS

3.1. Introduction

The Markov Chain Monte Carlo (MCMC) approach allows for a wide range of realistic statistical modeling. Until recently, recognizing the entire complexity and structure of many applications was challenging and necessitated the creation of specialized methods and technologies. The alternative was to force the problem into an overly simplistic framework of an existing approach. MCMC approaches now provide an unified framework for the analysis of many difficult issues utilizing generic software (Gilks et al. 1996). In this chapter we showed a general overview about Monte Carlo technique. Then, we introduced the Monte Carlo study. Finally, the results and findings from this technique.

3.2. Overview

Many scholars employ Monte Carlo simulation (MCS) techniques in their statistical and economic research. It is a very easy to implement method with a lot of benefits. General benefits or aims of MCS techniques are: First, assess an inference method's performance. Second, assess the parametric inference's resiliency to assumption violations. Compare the statistical qualities of different estimators in the third step. Finally, make conclusions when an estimator has a poor statistical theory. This technique was executed using R program. R is an open-source software and simple to learn. In the R programming language, libraries, also known as packages, play a vital role. It is made up of a number of statistical modeling algorithms and machine learning ideas that allow users to conduct repeatable research and develop informative materials. In order to generate Type II progressive censoring sample in R, we employed the same approach that was used in Balakrishnan and Sandhu Source (1995) following these steps:

1. Determine the values of n, m, μ, σ, α and R_i $(i = 1,2, \dots, m)$.

2. Create m independent units from Uniform (0,1) A_1, A_2,. .., A_m.
3. Define $W_i = A_i^{1/(i+R_m+R_{m-1}+\cdots+R_{m-i+1})}$, for i= 1,2,..,m.
4. Define U_i = 1- $W_m * W_{m-1} * \ldots * W_{m-i+1}$, for i = 1,2,..,m.

   Next U_1 , U_2 ,.., U_m is the needed progressive Type-II censored sample from the Uniform (0,1) distribution.
5. Lastly, set $X_i = F^{-1}(U_j)$, for i = 1, 2 ,..,m , where F^{-1} (.) is the inverse CDF of the distribution under study (Equation 10).

$$f(z|\alpha) = \frac{\alpha e^{-(z\alpha)}}{\left(1+e^{-(z)}\right)^{\alpha+1}}; \quad -\infty < z < \infty .$$

   Then X_1 , X_2 ,.., X_m is the required progressive Type-II censored sample from the distribution F(.) (Equation 11).

$$F(z|\alpha) = 1 - \left(\frac{e^{-(z)}}{\left(1+e^{-(z)}\right)}\right)^{\alpha}; \quad -\infty < z < \infty.$$

6. For a specific n, m and R (the scheme) , calculate the estimator (MLE , Bayesian : Lindley's approach or Importance Sampling BLUE or BLEE.
7. Repeat the previous steps K times, where K is the simulation times (5000 times).
8. Compute the Monte Carlo estimator, which is the average of the calculated estimators.
9. Compute the Bias and Mean Squared Error (MSE) of the Monte Carlo estimator.

   For each option of m and n, we executed the Monte Carlo simulation with various total sized n samples, with number of failures equals to m, and progressive censoring scheme options (CS).
10. Compare the MLE, Bayesian (Lindley's approach and Importance Sampling), BLUE and BLEE with respect to Bias and MSE.

In our study we used different schemes to find the estimators. The following table illustrate the progressive censoring schemes that we used in our study.

Table 1: Progressive Censoring Schemes Samples

	n	m	progressive censoring Scheme (CS)
1	50	30	(0*29,20)
2		30	(0*10,2*10,0*10)
3		30	(20,0*29)
4	50	40	(0*39,10)
5		40	(0*15,1*10,0*15)
6		40	(10,0*39)
7	70	40	(0*39,30)
8		40	(0*10,2*15,0*15)
9		40	(30,0*39)
10	70	50	(0*49,20)
11		50	(0*20,2*10,0*20)
12		50	(20,0*49)
13	90	50	(0*49,40)
14		50	(0*15,2*20,0*15)
15		50	(40,0*49)
16	90	60	(0*59,30)
17		60	(0*20,2*15,0*25)
18		60	(30,0*59)

3.3.Monte Carlo study

As a result, to run the simulation. We begin the data creation process. The parameters (μ , σ) are considered to have starting true values of (0,1), respectively. We assumed $\pi(\mu) = 1$ as a flat prior and the Jeffrey prior $\pi(\sigma) = 1/\sigma$ for location and scale parameters in terms of prior distributions. These are non-informative priors for the two parameters. Various sample sizes and progressive censoring strategies are used in Monte Carlo simulation research. Balakrishnan and Sandhu (1995), proposed a uniform variates-based approach for Type II progressive censored data distributed from Type II Generalized Logistic Distribution. The findings of the average values, Bias and MSEs of MLEs, Bayesian Lindley's approximation, Bayesian Importance Sampling estimators, BLUE and BLEE are presented in Tables 2, 3, 4, 5, 6 and 7. For sample sizes n=50, 70 and 90. Having smaller sample sizes will not give us efficient estimators, it might not give a defined value for an estimator. Observed failure periods m = 30, 40, 50 and 60, data from Type II Generalized Logistic Distribution with α =1.5 ,1, or 0.5 was used to simulate data. We calculated all the averages across 5000 simulations.

Table 2: Results of Simulation for Parameter μ with Generalized Logistic Distribution ($\alpha = 1.5$, $\mu = 0$, $\sigma = 1$)

	N	m	Scheme	MLE	Bayesian Lindley's	Bayesian Importance Sampling	BLUE	BLEE
	50	30	(0*29,20)					
Mean				-0.0316	-0.0411	-1.7436	0.0295	0.0101
Bias				-0.0316	-0.0411	-1.7436	0.0295	0.0101
MSE				0.0010	0.0017	3.0400	0.0660	0.0648
		30	(0*10,2*10,0*10)					
Mean				-0.0293	-0.0466	-1.3551	2.2187	2.1775
Bias				-0.0293	-0.0466	-1.3551	2.2187	2.1775
MSE				0.0009	0.0022	1.8362	4.9878	0.0648
		30	(20,0*29)					
Mean				-0.0092	-0.0929	-0.8390	2.6077	2.5681
Bias				-0.0092	-0.0929	-0.8390	2.6077	2.5681
MSE				0.0001	0.0086	0.7040	6.8653	0.0648
	50	40	(0*39,10)					
Mean				-0.0160	-0.0226	-1.2661	0.0172	0.0094
Bias				-0.0160	-0.0226	-1.2661	0.0172	0.0094
MSE				0.0003	0.0005	1.6030	0.0497	0.0493
		40	(0*15,1*10,0*15)					
Mean				-0.0137	-0.0421	-1.0062	0.9233	0.9108
Bias				-0.0137	-0.0421	-1.0062	0.9233	0.9108
MSE				0.0002	0.0018	1.0125	0.9019	0.0493
		40	(10,0*39)					
Mean				-0.0067	-0.0586	-0.7654	1.1288	1.1166
Bias				-0.0067	-0.0586	-0.7654	1.1288	1.1166
MSE				0.0000	0.0034	0.5858	1.3237	0.0493
	70	40	(0*39,30)					

Mean				-0.0246	-0.0294	-1.7559	0.0285	0.0129
Bias				-0.0246	-0.0294	-1.7559	0.0285	0.0129
MSE				0.0006	0.0009	3.0832	0.0506	0.0495
		40	(0*10,2*15,0*15)					
Mean				-0.0246	-0.0366	-1.2942	2.6859	2.6498
Bias				-0.0246	-0.0366	-1.2942	2.6859	2.6498
MSE				0.0006	0.0013	1.6750	7.2640	0.0495
	70	50	(0*49,20)					
Mean				-0.0147	-0.0224	-1.4289	0.0164	0.0085
Bias				-0.0147	-0.0224	-1.4289	0.0164	0.0085
MSE				0.0002	0.0005	2.0419	0.0389	0.0385
		50	(0*20,2*10,0*20)					
Mean				-0.0166	-0.0557	-1.0992	1.5217	1.5064
Bias				-0.0166	-0.0557	-1.0992	1.5217	1.5064
MSE				0.0003	0.0031	1.2083	2.3542	0.0385
		50	(20,0*49)					
Mean				-0.0101	-0.0557	-0.7403	1.8189	1.8040
Bias				-0.0101	-0.0557	-0.7403	1.8189	1.8040
MSE				0.0001	0.0031	0.5481	3.3470	0.0385
	90	50	(0*49,40)					
Mean				-0.0248	-0.0259	-1.7668	0.0183	0.0053
Bias				-0.0248	-0.0259	-1.7668	0.0183	0.0053
MSE				0.0006	0.0007	3.1217	0.0406	0.0401
		50	(0*15,2*20,0*15)					
Mean				-0.0153	-0.0312	-1.3673	2.8937	2.8620
Bias				-0.0153	-0.0312	-1.3673	2.8937	2.8620
MSE				0.0002	0.0010	1.8696	8.4135	0.0401
	90	60	(0*59,30)					
Mean				-0.0076	-0.0180	-1.5100	0.0143	0.0067

Bias			-0.0076	-0.0180	-1.5100	0.0143	0.0067
MSE			0.0001	0.0003	2.2800	0.0323	0.0321
	60	(0*20,2*15,0*25)					
Mean			-0.0067	-0.0252	-1.1241	2.0089	1.9925
Bias			-0.0067	-0.0252	-1.1241	2.0089	1.9925
MSE			0.0000	0.0006	1.2636	4.0679	0.0321
	60	(30,0*59)					
Mean			-0.0029	-0.0420	-0.7201	2.2792	2.2635
Bias			-0.0029	-0.0420	-0.7201	2.2792	2.2635
MSE			0.0000	0.0018	0.5185	5.2268	0.0321

In Table 2, the 5000 simulation executions were done to estimate the location parameter of Type II Generalized Logistic Distribution with α =1.5. We can notice that when n and m are fixed while changing the progressive censoring scheme, the MLE and Bayesian – Importance Sampling estimators are getting better values in terms of bias and MSE criteria. Comparing with the other estimators, the Bayesian – Lindley's Approach, BLUE and BLEE estimatots getter worse performance especially the BLUE.

For example, when n = 90 and m = 60 with progressive censoring schemes (0*59,30), (0*20,2*15,0*25) and (30,0*59), the MLE bias values are (-0.0076, -0.0067, -0.0029) , the BLUE bias values are (0.0143, 2.0089, 2.2792). Also, the MSE for BLEE is fixed when n and m are fixed. Moreover, such scheme (0*15,2*20,0*15) with n = 90 and m =50 has very bad influence on the BLUE estimator performace as well as the schema which not starting with zeros. To illustrate this, when having this scheme (0*59,30) with n= 90 and m = 60, the MSE for BLUE estimator is 0.0323, while when change the schema to (0*20,2*15,0*25) the BLUE MSE = 4.0679 .And with (30,0*59) scheme, the MSE of the BLUE = 5.2268.

Table 3: Results of Simulation for Parameter μ with Generalized Logistic Distribution (α =1.0, μ = 0, σ =1)

	N	m	Scheme	MLE	Bayesian Lindley's	Bayesian Importance Sampling	BLUE	BLEE
	50	30	(0*29,20)					
Mean				-0.0145	-0.0260	-1.2894	0.0078	-0.0010
Bias				-0.0145	-0.0260	-1.2894	0.0078	-0.0010
MSE				0.0002	0.0007	1.6625	0.0649	0.0648
		30	(0*10,2*10,0*10)					
Mean				-0.0223	-0.0400	-0.8053	1.8900	1.8698
Bias				-0.0223	-0.0400	-0.8053	1.8900	1.8698
MSE				0.0005	0.0016	0.6485	3.6369	0.0648
		30	(20,0*29)					
Mean				-0.0030	-0.0845	-0.2378	2.4078	2.3881
Bias				-0.0030	-0.0845	-0.2378	2.4078	2.3881
MSE				0.0000	0.0071	0.0565	5.8622	0.0648
	50	40	(0*39,10)					
Mean				-0.0044	-0.0148	-0.7395	-0.0040	-0.0056
Bias				-0.0044	-0.0148	-0.7395	-0.0040	-0.0056
MSE				0.0000	0.0002	0.5468	0.0584	0.0584
		40	(0*15,1*10,0*15)					
Mean				-0.0108	-0.0322	-0.4200	0.6519	0.6492
Bias				-0.0108	-0.0322	-0.4200	0.6519	0.6492
MSE				0.0001	0.0010	0.1764	0.4834	0.0584
		40	(10,0*39)					
Mean				0.0044	-0.0779	-0.1488	0.9265	0.9239
Bias				0.0044	-0.0779	-0.1488	0.9265	0.9239
MSE				0.0000	0.0061	0.0221	0.9169	0.0584

	70	40	(0*39,30)					
Mean				-0.0140	-0.0206	-1.3127	0.0046	-0.0028
Bias				-0.0140	-0.0206	-1.3127	0.0046	-0.0028
MSE				0.0002	0.0004	1.7231	0.0482	0.0482
		40	(0*10,2*15,0*15)					
Mean				-0.0094	-0.0276	-0.7473	2.3503	2.3314
Bias				-0.0094	-0.0276	-0.7473	2.3503	2.3314
MSE				0.0001	0.0008	0.5585	5.5720	0.0482
		40	(30,0*39)					
Mean				-0.0027	-0.0730	-0.1854	2.8241	2.8059
Bias				-0.0027	-0.0730	-0.1854	2.8241	2.8059
MSE				0.0000	0.0053	0.0344	8.0237	0.0482
	70	50	(0*49,20)					
Mean				-0.0020	-0.0148	-0.9359	-0.0020	-0.0045
Bias				-0.0020	-0.0148	-0.9359	-0.0020	-0.0045
MSE				0.0000	0.0002	0.8759	0.0432	0.0432
		50	(0*20,2*10,0*20)					
Mean				-0.0093	-0.0213	-0.5268	1.1800	1.1749
Bias				-0.0093	-0.0213	-0.5268	1.1800	1.1749
MSE				0.0001	0.0005	0.2775	1.4356	0.0432
		50	(20,0*49)					
Mean				-0.0081	-0.0561	-0.1273	1.5672	1.5622
Bias				-0.0081	-0.0561	-0.1273	1.5672	1.5622
MSE				0.0001	0.0032	0.0162	2.4993	0.0432
	90	50	(0*49,40)					
Mean				-0.0120	-0.0179	-1.3227	0.0062	-0.0002
Bias				-0.0120	-0.0179	-1.3227	0.0062	-0.0002
MSE				0.0001	0.0003	1.7496	0.0385	0.0384
		50	(0*15,2*20,0*15)					

Mean				-0.0150	-0.0156	-0.8236	2.5062	2.4892
Bias				-0.0150	-0.0156	-0.8236	2.5062	2.4892
MSE				0.0002	0.0002	0.6784	6.3193	0.0384
	90	60	(0*59,30)					
Mean				-0.0057	-0.0175	-1.0327	0.0018	-0.0010
Bias				-0.0057	-0.0175	-1.0327	0.0018	-0.0010
MSE				0.0000	0.0003	1.0664	0.0346	0.0346
		60	(0*20,2*15,0*25)					
Mean				-0.0045	-0.0221	-0.5478	1.6323	1.6258
Bias				-0.0045	-0.0221	-0.5478	1.6323	1.6258
MSE				0.0000	0.0005	0.3001	2.6990	0.0346
		60	(30,0*59)					
Mean				0.0012	-0.0510	-0.1158	2.0324	2.0260
Bias				0.0012	-0.0510	-0.1158	2.0324	2.0260
MSE				0.0000	0.0026	0.0134	4.1650	0.0346

In Table 3, the 5000 simulation executions were done to estimate the location parameter of Type II Generalized Logistic Distribution with α =1.0. We can notice the same behaviour of the estimators as when α =1.5, with a slight increase in bias and MSE values for MLE estimator when testing with this scheme (0*20,2*10,0*20). Also, the decrease in the α values will enhance the performance of the MLE. For instant, the bias of MLE when n = 90 and m=60 will be (-0.0076, -0.0076, -0.0029) for different values of schemes with α =1.5. on the other hand, it will be (-0.0057, -0.0045,0.0012) for different values of schemes with α =1.0. The same improvement in the performance with all other estimators when α decreases (α =1.0). For the Bayesian – Lindley's Approach, when the scheme is (30,0*59) the performance getting worse than with α =1.5. While the Bayesian – Importance Sampling estimator has an immense improvement in the performance when α is decreasing.

Table 4: Results of Simulation for Parameter μ with Generalized Logistic Distribution (α =0.5,μ = 0, σ =1)

	N	m	Scheme	MLE	Bayesian Lindley's	Bayesian Importance Sampling	BLUE	BLEE
	50	30	(0*29,20)					
Mean				0.0155	-0.0507	-0.3528	-0.0283	-0.0219
Bias				0.0155	-0.0507	-0.3528	-0.0283	-0.0219
MSE				0.0002	0.0026	0.1245	0.0997	0.0989
		30	(0*10,2*10,0*10)					
Mean				-0.0015	-0.0836	0.3704	0.8626	0.8792
Bias				-0.0015	-0.0836	0.3704	0.8626	0.8792
MSE				0.0000	0.0070	0.1372	0.8430	0.0989
		30	(20,0*29)					
Mean				0.0007	-0.2832	1.1404	1.6587	1.6758
Bias				0.0007	-0.2832	1.1404	1.6587	1.6758
MSE				0.0000	0.0802	1.3005	2.8502	0.0989
	50	40	(0*39,10)					
Mean				0.0140	-0.0257	0.3215	-0.0389	-0.0319
Bias				0.0140	-0.0257	0.3215	-0.0389	-0.0319
MSE				0.0002	0.0007	0.1033	0.1003	0.0987
		40	(0*15,1*10,0*15)					
Mean				0.0081	-0.1002	0.8464	0.0444	0.0564
Bias				0.0081	-0.1002	0.8464	0.0444	0.0564
MSE				0.0001	0.0100	0.7164	0.1007	0.0987
		40	(10,0*39)					
Mean				0.0062	-0.2277	1.2132	0.4070	0.4193
Bias				0.0062	-0.2277	1.2132	0.4070	0.4193
MSE				0.0000	0.0519	1.4719	0.2644	0.0987
	70	40	(0*39,30)					

Mean				0.0072	-0.0312	-0.4076	-0.0225	-0.0183
Bias				0.0072	-0.0312	-0.4076	-0.0225	-0.0183
MSE				0.0001	0.0010	0.1661	0.0720	0.0715
		40	(0*10,2*15,0*15)					
Mean				-0.0026	-0.0649	0.4517	1.2506	1.2631
Bias				-0.0026	-0.0649	0.4517	1.2506	1.2631
MSE				0.0000	0.0042	0.2040	1.6354	0.0715
		40	(30,0*39)					
Mean				0.0013	-0.2201	1.1894	2.0300	2.0426
Bias				0.0013	-0.2201	1.1894	2.0300	2.0426
MSE				0.0000	0.0484	1.4147	4.1924	0.0715
	70	50	(0*49,20)					
Mean				0.0022	-0.0221	0.0621	-0.0313	-0.0263
Bias				0.0022	-0.0221	0.0621	-0.0313	-0.0263
MSE				0.0000	0.0005	0.0039	0.0723	0.0713
		50	(0*20,2*10,0*20)					
Mean				0.0092	-0.0650	0.7188	0.3066	0.3177
Bias				0.0092	-0.0650	0.7188	0.3066	0.3177
MSE				0.0001	0.0042	0.5167	0.1653	0.0713
		50	(20,0*49)					
Mean				0.0082	-0.1819	1.2491	0.8419	0.8532
Bias				0.0082	-0.1819	1.2491	0.8419	0.8532
MSE				0.0001	0.0331	1.5603	0.7801	0.0713
	90	50	(0*49,40)					
Mean				0.0094	-0.0294	-0.4368	-0.0169	-0.0138
Bias				0.0094	-0.0294	-0.4368	-0.0169	-0.0138
MSE				0.0001	0.0009	0.1908	0.0563	0.0560
		50	(0*15,2*20,0*15)					
Mean				0.0023	-0.0443	0.3366	1.3371	1.3468

Bias				0.0023	-0.0443	0.3366	1.3371	1.3468
MSE				0.0000	0.0020	0.1133	1.8439	0.0560
		50	(40,0*49)					
Mean				0.0066	-0.1864	1.2254	2.2811	2.2910
Bias				0.0066	-0.1864	1.2254	2.2811	2.2910
MSE				0.0000	0.0348	1.5017	5.2593	0.0560
	90	60	(0*59,30)					
Mean				0.0086	-0.0152	-0.0725	-0.0217	-0.0178
Bias				0.0086	-0.0152	-0.0725	-0.0217	-0.0178
MSE				0.0001	0.0002	0.0053	0.0563	0.0558
		60	(0*20,2*15,0*25)					
Mean				0.0041	-0.0531	0.6870	0.5890	0.5989
Bias				0.0041	-0.0531	0.6870	0.5890	0.5989
MSE				0.0000	0.0028	0.4719	0.4027	0.0558
		60	(30,0*59)					
Mean				0.0071	-0.1501	1.2685	1.1942	1.2042
Bias				0.0071	-0.1501	1.2685	1.1942	1.2042
MSE				0.0001	0.0225	1.6090	1.4820	0.0558

In Table 4, the 5000 simulation executions were done to estimate the location parameter of Type II Generalized Logistic Distribution with α =0.5. The performance of the estimators MLE is getting better with different schemes while n and m are fixed. The acting of the estimators are the same as before , the performace is getting worse with Bayesian – Lindley's Approach , BLUE and BLEE . On the contrary, the behaviour of the Bayesian – Importance Sampling is getting really worse when α =0.5. Overall performance of all estimators when α =0.5 is worse than the one when α =1.0 and 1.5, except for the BLUE.

Table 5: Results of Simulation for Parameter σ with Generalized Logistic Distribution (α =1.5, μ = 0, σ =1)

	N	m	Scheme	MLE	Bayesian Lindley's	Bayesian Importance Sampling	BLUE	BLEE
	50	30	(0*29,20)					
Mean				0.9711	0.9991	1.3606	1.0558	1.0291
Bias				-0.0289	-0.0009	0.3606	0.0558	0.0291
MSE				0.0008	0.0000	0.1300	0.0290	0.0253
		30	(0*10,2*10,0*10)					
Mean				0.9789	0.9931	1.0971	2.2428	2.1861
Bias				-0.0211	-0.0069	0.0971	1.2428	1.1861
MSE				0.0004	0.0000	0.0094	1.5704	0.0253
		30	(20,0*29)					
Mean				0.9846	1.0060	1.0508	2.1522	2.0979
Bias				-0.0154	0.0060	0.0508	1.1522	1.0979
MSE				0.0002	0.0000	0.0026	1.3535	0.0253
	50	40	(0*39,10)					
Mean				0.9810	1.0063	1.1550	1.0460	1.0278
Bias				-0.0190	0.0063	0.1550	0.0460	0.0278
MSE				0.0004	0.0000	0.0240	0.0198	0.0174
		40	(0*15,1*10,0*15)					
Mean				0.9848	1.0001	1.0689	1.6908	0.6614
Bias				-0.0152	0.0001	0.0689	0.6908	0.6614
MSE				0.0002	0.0000	0.0047	0.4949	0.0174
		40	(10,0*39)					
Mean				0.9866	1.0010	1.0526	1.6559	1.6272
Bias				-0.0134	0.0010	0.0526	0.6559	0.6272
MSE				0.0002	0.0000	0.0028	0.4479	0.0174
	70	40	(0*39,30)					

Mean			0.9811	0.9957	1.3667	1.0448	1.0247
Bias			-0.0189	-0.0043	0.3667	0.0448	0.0247
MSE			0.0004	0.0000	0.1345	0.0216	0.0192
		40	(0*10,2*15,0*15)				
Mean			0.9846	0.9983	1.0614	2.4195	2.3730
Bias			-0.0154	-0.0017	0.0614	1.4195	1.3730
MSE			0.0002	0.0000	0.0038	2.0347	0.0192
	70	50	(0*49,20)				
Mean			0.9847	1.0000	1.2044	1.0359	1.0210
Bias			-0.0153	0.0000	0.2044	0.0359	0.0210
MSE			0.0002	0.0000	0.0418	0.0159	0.0144
		50	(0*20,2*10,0*20)				
Mean			0.9874	1.0015	1.0639	1.9904	1.9617
Bias			-0.0126	0.0015	0.0639	0.9904	0.9617
MSE			0.0002	0.0000	0.0041	0.9955	0.0144
		50	(20,0*49)				
Mean			0.9900	1.0015	1.0413	1.9326	1.9047
Bias			-0.0100	0.0015	0.0413	0.9326	0.9047
MSE			0.0001	0.0000	0.0017	0.8843	0.0144
	90	50	(0*49,40)				
Mean			0.9822	0.9975	1.3658	1.0389	1.0228
Bias			-0.0178	-0.0025	0.3658	0.0389	0.0228
MSE			0.0003	0.0000	0.1338	0.0173	0.0228
		50	(0*15,2*20,0*15)				
Mean			0.9892	0.9938	1.0843	2.5284	2.4892
Bias			-0.0108	-0.0062	0.0843	1.5284	1.4892
MSE			0.0001	0.0000	0.0071	2.3518	0.0155
	90	60	(0*59,30)				
Mean			0.9885	0.9992	1.2394	1.0315	1.0188

Bias			-0.0115	-0.0008	0.2394	0.0315	0.0188
MSE			0.0001	0.0000	0.0573	0.0134	0.0123
	60	(0*20,2*15,0*25)					
Mean			0.9908	0.9994	1.0529	2.2133	2.1860
Bias			-0.0092	-0.0006	0.0529	1.2133	1.1860
MSE			0.0001	0.0000	0.0028	1.4845	0.0123
	60	(30,0*59)					
Mean			0.9879	1.0044	1.0405	2.1111	2.0851
Bias			-0.0121	0.0044	0.0405	1.1111	1.0851
MSE			0.0001	0.0000	0.0016	1.2469	0.0123

In Table 5, the 5000 simulation executions were done to estimate the scale parameter of Type II Generalized Logistic Distribution with α =1.5. We can notice that when n and m are fixed while changing the progressive censoring scheme, the MLE and Bayesian – Importance Sampling estimators are getting better values in terms of bias and MSE criteria. Comparing with the other estimators, the Bayesian – Lindley's Approach, BLUE and BLEE estimatots getter worse performance especially the BLUE. There is a strage behaviour when the scheme is (0*20,2*15,0*25), if the estimator is decreasing in the case of this scheme, the estimator is increasing and vise versa.

For example, when n = 50 and m = 30 with progressive censoring scheme (0*29,20), (0*10,2*10,0*10) and (20,0*29) , MLE bias (-0.0289, -0.0211, -0.0154) , BLUE bias (0.0558, 1.2428, 1.1522). Also, the MSE for BLEE is fixed when n and m are fixed. Moreover, such scheme (0*15,2*20,0*15) has very weired bahaviour. As we can see this act from the bias for BLUE started with 0.0558 and ended with 1.1522 which show the increase in the value, but having the value 1.2428 with this scheme.

Table 6: Results of Simulation for Parameter σ with Generalized Logistic Distribution (α =1.0,μ = 0, σ =1)

	N	m	Scheme	MLE	Bayesian Lindley's	Bayesian Importance Sampling	BLUE	BLEE
	50	30	(0*29,20)					
Mean				0.9744	1.0105	1.1913	1.0559	1.0298
Bias				-0.0256	0.0105	0.1913	0.0559	0.0298
MSE				0.0007	0.0001	0.0366	0.0285	0.0247
		30	(0*10,2*10,0*10)					
Mean				0.9811	0.9985	1.0560	2.4334	2.3733
Bias				-0.0189	-0.0015	0.0560	1.4334	1.3733
MSE				0.0004	0.0000	0.0031	2.0801	0.0247
		30	(20,0*29)					
Mean				0.9856	0.9951	1.0532	2.3737	2.3151
Bias				-0.0144	-0.0049	0.0532	1.3737	1.3151
MSE				0.0002	0.0000	0.0028	1.9125	0.0247
	50	40	(0*39,10)					
Mean				0.9850	1.0064	1.0746	1.0473	1.0292
Bias				-0.0150	0.0064	0.0746	0.0473	0.0292
MSE				0.0002	0.0000	0.0056	0.0199	0.0173
		40	(0*15,1*10,0*15)					
Mean				0.9840	1.0019	1.0416	1.7485	1.7182
Bias				-0.0160	0.0019	0.0416	0.7485	0.7182
MSE				0.0003	0.0000	0.0017	0.5779	0.0173
		40	(10,0*39)					
Mean				0.9897	0.9987	1.0398	1.7399	1.7098
Bias				-0.0103	-0.0013	0.0398	0.7399	0.7098
MSE				0.0001	0.0000	0.0016	0.5651	0.0173
	70	40	(0*39,30)					

Mean			0.9827	1.0067	1.1925	1.0424	1.0228
Bias			-0.0173	0.0067	0.1925	0.0424	0.0228
MSE			0.0003	0.0000	0.0371	0.0209	0.0188
		40	(0*10,2*15,0*15)				
Mean			0.9839	0.9985	1.0332	2.6443	2.5946
Bias			-0.0161	-0.0015	0.0332	1.6443	1.5946
MSE			0.0003	0.0000	0.0011	2.7228	0.0188
		40	(30,0*39)				
Mean			0.9909	0.9997	1.0343	2.5484	2.5005
Bias			-0.0091	-0.0003	0.0343	1.5484	1.5005
MSE			0.0001	0.0000	0.0012	2.4167	0.0188
	70	50	(0*49,20)				
Mean			0.9870	1.0095	1.0982	1.0349	1.0202
Bias			-0.0130	0.0095	0.0982	0.0349	0.0202
MSE			0.0002	0.0001	0.0096	0.0157	0.0142
		50	(0*20,2*10,0*20)				
Mean			0.9885	1.0011	1.0292	2.1164	2.0863
Bias			-0.0115	0.0011	0.0292	1.1164	1.0863
MSE			0.0001	0.0000	0.0009	1.2608	0.0142
		50	(20,0*49)				
Mean			0.9912	0.9992	1.0325	2.0805	2.0509
Bias			-0.0088	-0.0008	0.0325	1.0805	1.0509
MSE			0.0001	0.0000	0.0011	1.1820	0.0142
	90	50	(0*49,40)				
Mean			0.9851	1.0006	1.1943	1.0357	1.0200
Bias			-0.0149	0.0006	0.1943	0.0357	0.0200
MSE			0.0002	0.0000	0.0378	0.0167	0.0152
		50	(0*15,2*20,0*15)				
Mean			0.9871	1.0017	1.0374	2.7541	2.7123

Bias				-0.0129	0.0017	0.0374	1.7541	1.7123
MSE				0.0002	0.0000	0.0014	3.0922	0.0152
	90	60	(0*59,30)					
Mean				0.9874	1.0030	1.1154	1.0308	1.0183
Bias				-0.0126	0.0030	0.1154	0.0308	0.0183
MSE				0.0002	0.0000	0.0133	0.0132	0.0121
		60	(0*20,2*15,0*25)					
Mean				0.9900	0.9993	1.0269	2.3707	2.3420
Bias				-0.0100	-0.0007	0.0269	1.3707	1.3420
MSE				0.0001	0.0000	0.0007	1.8909	0.0121
		60	(30,0*59)					
Mean				0.9919	1.0004	1.0262	2.3090	2.2812
Bias				-0.0081	0.0004	0.0262	1.3090	1.2812
MSE				0.0001	0.0000	0.0007	1.7258	0.0121

In Table 6, the 5000 simulation executions were done to estimate the scale parameter of Type II Generalized Logistic Distribution with α =1.0. We can notice the same behaviour of the estimators as when α =1.5, with a slight increase in bias and MSE values for MLE estimator when testing with this scheme (0*20,2*10,0*20). Also, the decrease in the α values will enhance the performance of the MLE. For instance, the bias of MLE when n = 50 and m=30 will be (-0.0289, -0.0211, -0.0154) for different values of schemes with α =1.5. on the other hand, it will be (-0.0256, -0.0189, -0.0144) for different values of schemes with α =1.0. The same improvement in the performance with all other estimators when α decreases (α =1.0). For the Bayesian – Importance Sampling, BLUE and BLEE estimators act in a different way when using the (0*20,2*10,0*20) scheme.

Table 7: Results of Simulation for Parameter σ with Generalized Logistic Distribution (α =0.5,μ = 0, σ =1)

	N	m	Scheme	MLE	Bayesian Lindley's	Bayesian Importance Sampling	BLUE	BLEE
	50	30	(0*29,20)					
Mean				0.9794	1.0537	1.0684	1.0528	1.0274
Bias				-0.0206	0.0537	0.0684	0.0528	1.0274
MSE				0.0004	0.0029	0.0047	0.0275	0.0241
		30	(0*10,2*10,0*10)					
Mean				0.9830	0.9995	1.0779	2.7266	2.6609
Bias				-0.0170	-0.0005	0.0779	1.7266	1.6609
MSE				0.0003	0.0000	0.0061	3.0060	0.0241
		30	(20,0*29)					
Mean				0.9849	0.9940	1.1052	2.8265	2.7584
Bias				-0.0151	-0.0060	0.1052	1.8265	1.7584
MSE				0.0002	0.0000	0.0111	3.3607	0.0241
	50	40	(0*39,10)					
Mean				0.9876	1.0022	1.0422	1.0506	1.0318
Bias				-0.0124	0.0022	0.0422	0.0506	-0.0319
MSE				0.0002	0.0000	0.0018	0.0208	0.0179
		40	(0*15,1*10,0*15)					
Mean				0.9831	1.0018	1.0696	1.8021	1.7697
Bias				-0.0169	0.0018	0.0696	0.8021	0.7697
MSE				0.0003	0.0000	0.0048	0.6616	0.0179
		40	(10,0*39)					
Mean				0.9868	0.9929	1.0963	1.8504	1.8172
Bias				-0.0132	-0.0071	0.0963	0.8504	0.8172
MSE				0.0002	0.0001	0.0093	0.7414	0.0179
	70	40	(0*39,30)					

Mean				0.9811	1.0416	1.0590	1.0466	1.0275
Bias				-0.0189	0.0416	0.0590	0.0466	0.0275
MSE				0.0004	0.0017	0.0035	0.0207	0.0182
		40	(0*10,2*15,0*15)					
Mean				0.9860	0.9983	1.0670	3.0821	3.0260
Bias				-0.0140	-0.0017	0.0670	2.0821	2.0260
MSE				0.0002	0.0000	0.0045	4.3539	0.0182
		40	(30,0*39)					
Mean				0.9879	0.9915	1.0948	3.1116	3.0549
Bias				-0.0121	-0.0085	0.0948	2.1116	2.0549
MSE				0.0001	0.0001	0.0090	4.4772	0.0182
	70	50	(0*49,20)					
Mean				0.9907	1.0114	1.0332	1.0383	1.0234
Bias				-0.0093	0.0114	0.0332	0.0383	0.0234
MSE				0.0001	0.0001	0.0011	0.0160	0.0143
		50	(0*20,2*10,0*20)					
Mean				0.9887	1.0030	1.0657	2.2792	2.2465
Bias				-0.0113	0.0030	0.0657	1.2792	1.2465
MSE				0.0001	0.0000	0.0043	1.6509	0.0143
		50	(20,0*49)					
Mean				0.9894	0.9911	1.0832	2.3285	2.2951
Bias				-0.0106	-0.0089	0.0832	1.3285	1.2951
MSE				0.0001	0.0001	0.0069	1.7796	0.0143
	90	50	(0*49,40)					
Mean				0.9854	1.0334	1.0548	1.0354	1.0202
Bias				-0.0146	0.0334	0.0548	0.0354	0.0202
MSE				0.0002	0.0011	0.0030	0.0161	0.0147
		50	(0*15,2*20,0*15)					
Mean				0.9866	0.9999	1.0459	3.2139	3.1669

Bias			-0.0134	-0.0001	0.0459	2.2139	2.1669
MSE			0.0002	0.0000	0.0021	4.9164	0.0147
		50 (40,0*49)					
Mean			0.9919	0.9970	1.0860	3.3172	3.2686
Bias			-0.0081	-0.0030	0.0860	2.3172	2.2686
MSE			0.0001	0.0000	0.0074	5.3844	0.0147
	90	60 (0*59,30)					
Mean			0.9879	1.0179	1.0277	1.0312	1.0188
Bias			-0.0121	0.0179	0.0277	0.0312	0.0188
MSE			0.0001	0.0003	0.0008	0.0131	0.0120
		60 (0*20,2*15,0*25)					
Mean			0.9924	0.9992	1.0602	2.6402	2.6085
Bias			-0.0076	-0.0008	0.0602	1.6402	1.6085
MSE			0.0001	0.0000	0.0036	2.7023	0.0120
		60 (30,0*59)					
Mean			0.9912	0.9964	1.0773	2.6694	2.6373
Bias			-0.0088	-0.0036	0.0773	1.6694	1.6373
MSE			0.0001	0.0000	0.0060	2.7990	0.0120

In Table 7, the 5000 simulation executions were done to estimate the scale parameter of Type II Generalized Logistic Distribution with α =0.5. The performance of the estimators MLE is getting better with different schemes while n and m are fixed. The acting of the estimators are the same as before , but the performace is getting worse almost for all estimators.

3.4. Results and Findings

Because of their flexibility in data processing, generalized distributions have become popular in applications. The Generalized Logistic Distribution and its several forms, in particular, have recently gained a lot of interest.

We looked at point estimate of location and scale parameters of Type II Generalized Logistic Distribution in this study, which was under Type II progressively censored sample. Maximum likelihood estimators, Bayes estimators (Lindley's approximation and Importance Sampling estimation), Best Linear Unbiased Estimator, and Best (Affine) Linear Equivariant Estimator were created to inference the unknown parameters. Using bias and mean squared error as the criteria of comparison. the estimators were created and tested using simulation with different sample sizes with several progressive censoring schemes. In general, the results of a simulation study with Generalized Logistic Distribution (α =1.5, μ = 0, , σ =1), Generalized Logistic Distribution (α =1.0, μ = 0, , σ =1) and Generalized Logistic Distribution (α =0.5, μ = 0, , σ =1) reveal the following:

- When estimating the location parameter μ:
    - MLE is the most efficient one. The Lindley's approximation (Bayesian) is the one that comes closest to MLE. Then BLEE after that BLUE. The worst estimator is the Importance Sampling.
    - As n increases, the estimators getting better with respect to smallest Bias and MSE, especially for MLE. Again, MLE is the most efficient estimator then Lindley's approximation (Bayesian), BLEE after that BLUE. The worst estimator is the Importance Sampling.

- The Importance Sampling estimator is getting worse in term of Bias and MSE when the progressive scheme ends with zeros.
- As α decreases, same efficiency order of the estimators. Also, the Importance Sampling estimator getting better.

- When estimating the scale parameter σ :
  - The Lindley's approximation (Bayesian) is the most efficient estimator. MLE almost as efficient as the Lindley's. Then Importance Sampling then BLEE. The worst estimator is the BLUE.
  - As n increases, the estimators getting better with respect to smallest Bias and MSE.
  - When the progressive scheme ends with zeros, the Importance Sampling estimator improves in term of Bias and MSE while the BLUE getting worse.
  - As α decreases, same efficiency order of the estimators with slight improves in values with respect to bias and MSE.
  - Comparing the BLUE and BLEE estimators, the BLEE is more efficient than the BLUE. But Importance Sampling estimators is better than both BLUE and BLEE.

CHAPTER 4: APPLICATIONS

4.1. Introduction

In this chapter we applied different actual datasets, that fitted the Type II Generalized Logistic Distribution, extracted from real-life situations. First case was about strength of single carbon fibers. The experiment is based on a non-censoring sample. Second one, was about breakdown of an insulating fluid under Type II Progressive Censoring. Finally, we had a dataset representing the measurements of a certain characteristic in blood cells. Finally, the real data summary.

4.1. Case 1: Strength of Single Carbon Fibers (Complete Sample)

Carbon fiber is a material made up of tiny, strong carbon crystalline threads. The fibers are stiff, robust, and light, and are utilized in a variety of procedures to make high-quality construction materials. Carbon fiber has many capabilities, including high rigidity, high tensile strength, low weight-to-strength ratio, high temperature endurance, low thermal expansion, and strong chemical resistance.

We present the analysis of one actual data set for demonstration purposes. Badar and Priest (1982) were the first to examine because of the strengthen of the data. The evaluation of data is in GPA. The data reflects the strength of single carbon fibers and soaked 1000-carbon fiber tows. Individual fibers were stressed at gauge lengths of 1, 10, 20, and 50 mm. At gauge lengths of 20, 50, 150, and 300 mm, 1000 fiber-soaked tows were examined. For illustrative purposes, we'll utilize the single fibers data set of 10 mms in gauge lengths with sample size 63.

Many authors examined and analyzed this data set in their research and articles, for example: Gupta and Kundu (2010), Asgharzadeh, Esmaeili, Nadarajah and Shih (2013), Sindhu, Aslam and Hussain (2016), Femi Samuel Adeyinka (2019). Referring to Asgharzadeh, Esmaeili, Nadarajah and Shih (2013) study, they analyzed the data set using Kolmogorov–Smirnov statistic and calculated the p-value. They found K-S test to be 0.097 and the p-value = 0.571 which proves that the data fitted Generalized Logistic Distribution.

Table 8: Strength of Single Carbon Fibers Data Set

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397,
2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614,
2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917,
2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145,
3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346,
3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628,
3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Table 9: Results for μ Estimators Comparison

μ	**Estimator**	**Bias**	**MSE**
MLE	3.0245	3.0245	9.1473
Bayesian – Lindley's Approach	3.0074	3.0074	9.0441
Bayesian –Importance Sampling	3.0194	3.0194	9.1168
BLUE	3.0229	3.0229	9.1843
BLEE	3.0229	3.0229	0.0462

Table 10: Results for σ Estimators Comparisonsable

σ	Estimator	Bias	MSE
MLE	0.3525	-0.6475	0.4193
Bayesian – Lindley's Approach	0.3554	-0.6446	0.4155
Bayesian – Importance Sampling	0.8102	-0.1898	0.0360
BLUE	0.3711	-0.6289	0.4065
BLEE	0.3671	-0.6329	0.0109

Obviously, with the previous the assumptions, the BLEE estimators are the most efficient among the other estimators with MSE (0.0462,0.0109) for both parameters, location, and scale. It is clear that with complete censoring sample, the results are not so efficient as with the progressive censoring data.

Case 2: Breakdown of an Insulating Fluid (Type II Progressive Censoring)

To evaluate and analyze the quality of transformers and their insulating fluids, a variety of tests have been devised. To explain this, for example, let's consider the Dielectric Breakdown Test, which assesses an insulating liquid's capacity to endure electrical stress up to the point of failure. It displays the voltage at which there will be a breakdown. Moisture, dirt, and conductive particle contamination will induce failure at levels below what is considered tolerable.

Nelson (1982) provided a detailed progressively Type II censored data for the breakdown of an insulating fluid testing experiment. This data collection has been used by Viveros and Balakrishnan (1994). Moreover, it was examined and evaluated by Balakrishnan and Hossain (2007) examining Type II Generalized Logistic Distribution inference under progressive Type II censoring, which is the base of our research that we build on it to find the best estimator. Balakrishnan and Hossain evaluated and examined the data set that fits the Type II Generalized Logistic Distribution and finding out that MLE and Approximate MLE are very close in the inferencing. Also, Azizpour and Asgharzadeh (2018) studied this data set and they used the Kolmogorov-Smirnov (K-S) test to determine the validity of the Type-II Generalized Logistic Distribution with the associated p-values: K-S = 0.1399 and p-value = 0.8281. As a result, the Type-II Generalized Logistic Distribution fits the aforementioned data set fairly well.

In this example n= 19 and m=8 with α =1. The results are tabulated below:

Table 11: Breakdown of an Insulating Fluid Data Set

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
x_i	-1.6608	-0.2485	-0.0409	0.2700	1.0224	1.5789	1.8718	1.9947
r_i	0	0	3	0	3	0	0	5

Table 12: Results for μ Estimators Comparison

μ	**Estimator**	**Bias**	**MSE**
MLE	1.8757	1.8757	3.5184
Bayesian – Lindley's Approach	1.8511	1.8511	3.4266
Bayesian – Importance Sampling	-0.2370	-0.2370	0.0562
BLUE	2.5867	2.5867	2.5867
BLEE	2.4809	2.4809	0.2377

Table 13: Results for σ Estimators Comparisons

σ	Estimator	Bias	MSE
MLE	0.9027	-0.0973	0.0095
Bayesian – Lindley's Approach	1.8511	1.8511	3.4266
Bayesian – Importance Sampling	1.4455	0.4455	0.1985
BLUE	1.4211	0.4211	0.2887
BLEE	1.2786	0.2786	0.1003

From the results above, we conclude that with the previous assumptions, the Bayesian – Importance Sampling estimator is the most efficient among the other estimators for estimating the location parameter with MSE 0.0562. The MLEs almost as efficient as the Bayesian – Lindley's Approach when estimating the scale parameter with MSE (0.0095, 0.0008). Also, we can notice that these two estimators are very close in performance when estimating the location parameter as well.

4.2. Case 3: Type II Progressive Censoring Data

The following data set are part of 15 observations, from Tiku and Akkaya (2004), representing the measurements of a certain characteristic in blood cells. The given data set was fitted with the standard logistic distribution. The Kolmogorov-Smirnov (K-S) distances between the fitted and empirical distribution functions and the accompanying p value are 0.2155 0.4401 respectively. This shows that the standard logistic distribution (Type-II Generalized Logistic Distribution with $\alpha = 1$) fits the data rather well.

The following are the observational data and the progressive censoring scheme with sample of size $n = 15$ and $m=5$ distributed as Type II Generalized

Logistic Distribution with μ = 10.010, σ = 0.843 and α = 1.

Table 14: Type II Progressive Censoring Data Data Set

i	**1**	**2**	**3**	**4**	**5**
x_i	8.921	9.689	9.774	10.485	10.766
r_i	3	1	2	1	3

Table 15: Results for μ Estimators Comparison

μ	**Estimator**	**Bias**	**MSE**
MLE	10.6912	0.6812	0.4640
Bayesian – Lindley's Approach	10.6830	0.6730	0.4530
Bayesian – Importance Sampling	9.1055	-0.9045	0.8181
BLUE	-0.4328	-10.4428	121.2788
BLEE	1.4163	-8.5937	10.1868

Table 16: Results for σ Estimators Comparison

σ	**Estimator**	**Bias**	**MSE**
MLE	0.5168	-0.3262	0.1064
Bayesian – Lindley's Approach	0.5713	0.2717	0.0738
Bayesian – Importance Sampling	0.9997	0.1568	0.0246
BLUE	1.1968	0.3537	0.2694
BLEE	0.9949	0.1519	0.1199

Testing the results above, we can notice that the Bayesian – Lindley's Approximation as well as the MLE are the most efficient estimators for μ with MSE (0.4640, 0.4530) respectively. The worst estimator for μ is the BLUE. While the Bayesian estimator - Importance Sampling is the most efficient estimator for σ, close to it is the Lindley's approximation then the MLE followed by the BLEE. The worst of all is the BLUE. The MSEs of estimators are (0.0246, 0.0738, 0.1064, 0.1199, 0.2694) respectively. We can notice that the estimators of σ are close to each other's. The effect of small sample size is obvious on the other estimators especially the BLUE when estimating μ.

4.3. Real Data Summary:

As a brief of all the real data sets that we discussed earlier, it's crystal clear how the size of the sample affects the performance of the estimators especially with censored data, in specific the Bayesian – Importance Sampling, BLUE and BLEE estimators. Moreover, the type of the sample, whether a complete or censored one, has a critical impact on the efficiency of the inference of the estimators. Finally, the influence of progressive schemes for the censored data set is very crucial for the efficiency of the estimator.

CHAPTER 5: CONCLUSION AND FUTURE STUDY SUGGESTIONS

5.1. Introduction

This chapter will contain a research conclusion that analyzes the outcomes of the preceding chapters which focused on the point estimation of the unknown parameters of the type II generalized logistic distribution under type II progressive censoring sample. We evaluated the MLEs, Bayesian estimators and Linear estimators as different methods to do so based on mean squared error and bias criteria. To analyze and decide which is the most efficient estimator, we constructed a simulation method with 5000 iterations with different progressive schemes and sample sizes. Moreover, studied three different real data sets. The other part of this chapter is focused on some ideas and suggestions for further research that can be built on the current study.

5.2. Research Conclusion

Generalized distributions form a class of skewed distributions and gained widespread use in applications because of their flexibility in data analysis. More specifically, the Generalized Logistic Distribution with its different types has received considerable attention recently. In this study, based on progressively Type II censored data, we considered point estimation of location and scale parameters in Type II Generalized Logistic Distribution. We developed five estimators for its unknown parameters, including maximum likelihood estimators and Bayes estimators – Lindley's Approach and Importance Sampling, BLUE and BLEE. The estimators are compared using simulation based on the criteria of bias and Mean Squared Error.

Due to the non-closed form equations, the Newton–Raphson numerical method was used to find the ML estimator. Also, the Bayesian estimation were based on non-informative priors for both location and scale parameters.

Because of the intricacy of the ratio of two integrals, the Bayes estimations based on the squared error loss function (SELF) is complicated to be evaluated analytically. The Lindley's approach and Importance Sampling techniques were used to solve this situation. Furthermore, the BLUE and BLEE inference were considered as more estimators to investigate. Next, simulation research investigated the reality of the developed approaches for different value of sample sizes with n items, a sufficient failure cases m, and the alternative schemes of progressive censoring for each various alternative of n and m using a case of 5000. On the basis of a real-life example, the proposed methods were evaluated.

Results of this research reveal that MLE and Bayesian – Lindley's Approach are the most efficient estimators for location and scale in terms of bias and MSE. They have the smallest bias and MSE values as shown during the simulation process and real-life data experiments. Also, we noticed that the estimators are very sensitive to the progressive censoring schemes and sample size. Having a scheme with larger size. Moreover, conducting our study based on progressive censoring sampling gives better results than using complete censoring data. Finally, the effect of the α value on the estimator's bias and MSE values. We got better results when the value decreases.

5.3. Suggestions for Further Research

Due to time constraints, the following research ideas are recommended. First, using different methods to inference the unknown parameters of the location and scale of Type II Generalized Logistic Distribution, like confidence interval or testing methods. Secondly, the identical experiment might be run using a different loss function, namely the linear exponential (LINEX), balanced linex and balanced entropy loss functions, balanced squared error, and a different informative prior distribution with a variety of hyperparameters testing. Moreover, different type of progressive censoring data, for example Type I progressive censoring or hybrid progressive censoring. Also, further study could be conducted in order to establish the optimal progressive censorship scheme and redo the comparison with the same estimators. Finally, this estimation comparison could be tested on another type of the Generalized Logistic Distribution with different comparison criteria, like the mean, variance, or covariance ($\hat{\mu}$, $\hat{\sigma}$) of the estimator.

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