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Bifurcation of some new traveling wave solutions for the time–space M-fractional MEW equation via three altered methods

Imran Siddique [a], Khush Bukht Mehdi [a], Mohammed M.M. Jaradat [b,*], Asim Zafar [c], Mamdouh E. Elbrolosy [d,e], Adel A. Elmandouh [e,f], Mohammed Sallah [g,h]

[a] Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan
[b] Mathematics Program, Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, Doha P.O. Box 2713, Qatar
[c] Department of Mathematics, COMSATS University Islamabad, Vehari Campus, Pakistan
[d] Department of Mathematics and Statistics, College of Science, King Faisal University, P. O. Box 400, Al-Ahsa 31982, Saudi Arabia
[e] Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt
[f] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[g] Applied Mathematical Physics Research Group, Physics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[h] Higher Institute of Engineering and Technology, New Damietta, Egypt

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ABSTRACT

In this work,$(1/G^{'})$, modified $(G^{'}/G^2)$ and new extended direct algebraic methods are proposed to construct the novel exact traveling wave solutions in the form of trigonometric, hyperbolic and exponential functions of the time-fractional modified equal-width (MEW) equation in the sense of M- truncated fractional derivative. These methods contribute a variety of exact solutions in terms of the hyperbolic, trigonometric and rational functions to the scientific literature. The obtained solutions are verified for aforesaid equation through symbolic soft computations. To promote the essential propagated features, some investigated solutions are exhibited in the form of 2D and 3D graphics by passing on the precise values to the parameters under the constrain conditions. Further, the dynamical behavior is investigated. Based on the bifurcation constrains on the system's parameters, we constructed also some new wave solution which are assorted into solitary, kink, periodic, and super periodic wave solutions. The influence of the included parameters on the solution is clarified. Moreover, we guarantee that all the solutions are new and an excellent contribution in the existing literature of solitary wave theory.

Introduction

Nonlinear differential equations involving fractional order derivatives are general forms of integer order classical differential equations. It is well known that nonlinear fractional differential equations (FDEs) cover many fields such as physics, biomechanics, chemistry, biology, power-law non-locality, relativity, nonlinear optics, engineering, solid mechanics, electricity, signal processing and many others fields. Finding exact solutions of nonlinear (FDEs) is a very important part of nonlinear physical phenomenon. It is fact that exact solutions provide much physical information and help one to understand the mechanism that governs some physical models, such as plasma physics, optical fibers, biology, solid state physics, chemical physics, and so on. In recent years, many powerful methods for finding the exact solutions of nonlinear FDEs in the form of conformable fractional derivative [1], beta fraction derivative [2] and the M−fractional derivatives [3] have been proposed such as, exponential rational function method [4], Lie symmetry analysis method [5–7], Generalized (G'/G)- Expansion Method [8], Kudryashov method [9–11], residual power series method [12,13], sine-Gordon expansion method [14], extended sinh-Gordon equation expansion method [15–19], Hirota bilinear method [20], Riccati–Bernoulli sub-ODE technique [21], trigonometric function series method [22], auxiliary ordinary differential equation method [23], modified mapping method and the extended mapping method [24], modified trigonometric function series method [25], bifurcation method [26], modified (G'/G)-expansion method [27], extended (G/G')-expansion method [28], (G'/G)-expansion method [29,30], infinite series and cosine-function methods [31], generalized (G'/G)-expansion

* Corresponding author.
E-mail addresses: imransmsrazi@gmail.com (I. Siddique), bkmehdi786@gmail.com (K. Bukht Mehdi), mmjst4@qu.edu.qa (M.M.M. Jaradat), asimzafar@cuivehari.edu.pk (A. Zafar), melbrolosy@kfu.edu.sa (M.E. Elbrolosy), aelmandouh@kfu.edu.sa (A.A. Elmandouh), msallahd@mans.edu.eg (M. Sallah).

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method [32], tanh-coth expansion method [33], Jacobi elliptic function expansion method [34], first integral method [35], Variational principle method [36], Sardar-subequation method [37], new sub-equation method [38], extended direct algebraic method [39–41], exp $(-\phi(\eta))$ method [42] and Exp_a function method [43] and different other methods [44–47].

There is another important mathematical model named as: nonlinear MEW equation used for describing various fluid mechanics in nonlinear systems, plasma physics, and nonlinear optics [48]. This model has been solved to find the traveling wave solutions by many different effective methods. Pinar and Öziş [49] find the solutions of modified equal width equation by means of the auxiliary equation with a sixth degree nonlinear term. Application of the dynamical system method is shown in Su and Tang [50] to study the exact travelling wave solutions of the modified equal width equation. Solitary waves of modified equal width equation are presented by direct integration in Yang and Xu [51]. The extended simple equation method and the exp $(-\phi(\xi))$ expansion method are used for solving the modified equal width equation in Lu, et al. [52]. Raslan et al. [53] employed the modified extended tanh method with the Riccati equation for solving the space–time fractional modified equal width equation. Korkmaz implemented the various ansatz methods to construct the solutions of the space–time fractional modified equal width equation [54]. Lu and Ye [55] obtained the optical solitary wave solutions of the space–time fractional modified equal-width equation. Ali et al. [56] computed the traveling wave solution for nonlinear variable-order fractional model of modified equal width equation by using exp $(-\phi(\xi))$ method in the sense of Caputo fractional-order derivative.

The major concern of this existing study is to utilize the novel meanings of fractional-order derivative, named M−truncated fractional derivative, for pace-time fractional MEW equation, and to find the novel comprehensive exact traveling wave solutions in the form of hyperbolic, trigonometric and rational functions by employ-three modified methods, the $(1/G')-$ expansion method [57], the modified $(G'/G^2)-$ expansion method [58] and the new extended direct algebraic method [39–41]. To the best of our knowledge, the obtained solutions are more general and in different form which have never been reported in previously published studies [49–56]. Our results also enrich the variety of the dynamics of higher-dimensional non-linear wave field. It is hoped that these results will provide some valuable information in the higher-dimensional non-linear field.

Rest of the paper is structured as follows. In section 2, M−fractional derivative and its properties are presented, and the methodologies of our proposed three approaches are discussed in section 3. In section 4, exact traveling wave solutions of the time-fractional MEW equation are obtained via proposed methods. Section 5 includes the study of the bifurcation, constructing some wave solutions, and study the effect of the parameters on the wave solutions. In Section 6, the graphical comparisons of our obtained exact solutions are represented in both 2D and 3D plots for various values of parameters. At the end, conclusions are announced in *sec* 6.

The truncated M−Fractional derivative and its properties [3]:

In this section, we give the definition and properties of the truncated M−Fractional derivative of orderα.

Definition: Assume that $f:(0,\infty)\rightarrow R$, then, the new truncated M−fractional derivative of function f of order α is defined as,

$$D_M^{\alpha,\beta}f(t) = \lim_{\varepsilon\rightarrow 0}\frac{f\left(t\in_\beta\left(\varepsilon t^{1-\alpha}\right)\right)-f(t)}{\varepsilon},\ \text{for all}\ \ t>0,\ 0<\alpha<1,\ \beta>0,$$

where $\in_\beta(\cdot)$ is a truncated Mittag-Leffler function of one parameter [3].

Properties:

Let $\alpha\in(0,\ 1]$, $\beta>0$ and $f=f(t)$, $g=g(t)$ be α- differentiable, at a point $t>0$, then:

1. $D_M^{\alpha,\ \beta}(af+bg) = aD_M^{\alpha,\ \beta}f + bD_M^{\alpha,\ \beta}g$, for all$a,\ b\in R$..

$$D_M^{\alpha,\beta}(c)=0,\ \text{ where }\ f(t)=c,\ \text{ is a constant.}$$

$$D_M^{\alpha,\ \beta}(f\cdot g) = gD_M^{\alpha,\ \beta}f + fD_M^{\alpha,\ \beta}g,$$

$$D_M^{\alpha,\ \beta}\left(\frac{f}{g}\right) = \frac{gD_M^{\alpha,\ \beta}f - fD_M^{\alpha,\ \beta}g}{g^2},$$

Furthermore; if the function f is differentiable; then

$$D_M^{\alpha,\beta}f(t) = \frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{df}{dt}. \tag{1}$$

$$D_M^{\alpha,\beta}(f^\circ g)(t) = f'(g(t))D_M^{\alpha,\beta}g(t),\ \ \text{for}\ f\ \text{differentiable at}\ g(t).$$

This characterization also fulfills the Chain rule.

General form of the methods:

$(1/G')$- *expansion method*

Let us consider the nonlinear partial differential equation (NLPDE) is given by.

$$Q=(u,\ u_t,\ u_x,\ uu_x,\ u_{tt},\ Uu_{xt},\ uu_{xx},\ \ldots), \tag{2}$$

where $u=u(x,\ t)$ is an unknown function, Q is a polynomial depending on $u(x,t)$ and its various partial derivatives.

Step 1: By wave transformation.

$$\eta=(x-vt),\quad u(x,\ t)=U(\eta). \tag{3}$$

Here, v is the wave speed.

The wave variable allow us to reduce Eq. (3) into a nonlinear ordinary differential equation for$U=U(\eta)$:.

$$Q=(U,\ U',\ U'',\ U''',\ldots). \tag{4}$$

Step 2: Extend the solution of Eq. (4) in the following form.

$$U(\eta)=\sum_{i=0}^{m}a_i\left(\frac{1}{G'}\right)^i, \tag{5}$$

where and satisfies the following linear ordinary differential equation which is.

$$G''(\eta)+\lambda G'(\eta)+\mu=0, \tag{6}$$

where $a_i(i=1,\ldots,m),\lambda$ and μ are constants to be determined. The positive integer m can be obtained by using the homogenous balance between the highest order derivatives and the nonlinear term appearing in Eq. (4). Additionally, the solution of the differential equation given in Eq. (6) is.

$$G(\eta)=c_1e^{-\lambda\eta}-\frac{\mu\eta}{\lambda}+c_2, \tag{7}$$

where c_1 and c_2 are arbitrary integration constants. $(1/G')$ can be expressed as.

$$\left(\frac{1}{G'}\right)=\frac{\lambda}{-\mu+\lambda c_1[\cosh(\lambda\eta)-\sinh(\lambda\eta)]}. \tag{8}$$

Step 3: By substituting Eq. (5) into Eq. (4) and using Eq. (6), the left hand side of equation (4) can be converted into a polynomial in term of$(1/G^{'})$, equating each coefficient of the polynomial to zero yields a system of algebraic equations. By solving the algebraic equations with symbolic computation, we define$a_i(i=1,...,m),\lambda$ and μ..

The modified $(G^{'}/G^2)$- expansion method

Here, we will describe the basic steps of modified $(G^{'}/G^2)$- expansion method [58].

Step 1: Consider Eqs. (2), (3) and (4).

Step 2: Extend the solution of Eq. (4) in the following form.

$$U(\eta)=\sum_{i=0}^{m}a_i\left(\frac{G^{'}}{G^2}\right)^i, \tag{9}$$

where $a_i(i=0,1,2,3,...,m)$ are constants and find to be later. It is important that$a_i\neq 0$..

The function satisfies the following Riccati equation,

$$(\frac{G^{'}}{G^2})^{'}=\lambda_1\left(\frac{G^{'}}{G^2}\right)^2+\lambda_0, \tag{10}$$

where λ_0 and λ_1 are constants. We gain the below solutions to Eq. (10) due to different conditions ofλ_0:

When$\lambda_0\lambda_1<0$,.

$$\left(\frac{G^{'}}{G^2}\right)=-\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1}+\frac{\sqrt{|\lambda_0\lambda_1|}}{2}\left[\frac{c_1\sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+c_2\cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{c_1\cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+c_2\sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]. \tag{11}$$

When$\lambda_0\lambda_1>0$,.

$$\left(\frac{G^{'}}{G^2}\right)=\sqrt{\frac{\lambda_0}{\lambda_1}}\left[\frac{c_1\cos\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+c_2\sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{c_1\sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)-c_2\sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]. \tag{12}$$

When $\lambda_0=0$ and$\lambda_1\neq 0$,.

$$\left(\frac{G^{'}}{G^2}\right)=-\frac{c_1}{\lambda_1(c_1\eta+c_2)}. \tag{13}$$

where c_1 and c_2 are arbitrary constants.

Step 3: Replacing Eq. (9) into Eq. (3) along with Eq. (10) and tracing all coefficients of each $\left(\frac{G^{'}}{G^2}\right)^i$ to zero, then solving that algebraic equations generated in the term $a_i,\lambda_0,\lambda_1,c$ and other parameters.

Step 4: Replacing Eq. (9) of which α_i,ν and other parameters that are found in step 3 into Eq. (3), we get the solutions of Eq. (2).

The new extended direct algebraic method:

Here, we will describe the basic steps of new extended direct algebraic method or finding traveling wave solutions of nonlinear partial differential equations [39].

Step 1: Consider equations (2), (3) and (4).

Step 2: Extend the solution of equation (4) in the following form.

$$U(\eta)=a_0+\sum_{i=1}^{m}[a_iW(\eta)], \tag{14}$$

where,

$$W^{'}(\eta)=\ln(\rho)\left(\mu+\nu W(\eta)+\zeta W^2(\eta)\right),\ \rho\neq 0,1. \tag{15}$$

Here ν,μ along with ζ are the real constants which can be evaluated by balancing the highest order derivative along with nonlinear terms of Eq. (4).

General solutions of Eq. (15) as regards with parameters ν,μ and ζ are as follows [39].

$$\phi=\nu^2-4\mu\zeta.$$

(I) : If $\phi<0$ and $\zeta\neq 0$,

$$W_1(\eta)=-\frac{\nu}{2\zeta}+\frac{\sqrt{-\phi}}{2\zeta}\tan_\rho\left(\frac{\sqrt{-\phi}}{2}\eta\right), \tag{16}$$

$$W_2(\eta)=-\frac{\nu}{2\zeta}-\frac{\sqrt{-\phi}}{2\zeta}\cot_\rho\left(\frac{\sqrt{-\phi}}{2}\eta\right), \tag{17}$$

$$W_3(\eta)=-\frac{\nu}{2\zeta}+\frac{\sqrt{-\phi}}{2\zeta}\left(\tan_\rho\left(\sqrt{-\phi}\,\eta\right)\pm\sqrt{mn}\sec_\rho\left(\sqrt{-\phi}\,\eta\right)\right), \tag{18}$$

$$W_4(\eta)=-\frac{\nu}{2\zeta}+\frac{\sqrt{-\phi}}{2\zeta}\left(\cot_\rho\left(\sqrt{-\phi}\,\eta\right)\pm\sqrt{mn}\csc_\rho\left(\sqrt{-\phi}\,\eta\right)\right), \tag{19}$$

$$W_5(\eta)=-\frac{\nu}{2\zeta}+\frac{\sqrt{-\phi}}{4\zeta}\left(\tan_\rho\left(\frac{\sqrt{-\phi}}{4}\eta\right)-\cot_\rho\left(\frac{\sqrt{-\phi}}{4}\eta\right)\right).(II)$$

: If $\phi>0$ and $\zeta\neq 0$, (20)

$$W_6(\eta)=-\frac{\nu}{2\zeta}-\frac{\sqrt{\phi}}{2\zeta}\tanh_\rho\left(\frac{\sqrt{\phi}}{2}\eta\right), \tag{21}$$

$$W_7(\eta)=-\frac{\nu}{2\zeta}-\frac{\sqrt{\phi}}{2\zeta}\coth_\rho\left(\frac{\sqrt{\phi}}{2}\eta\right), \tag{22}$$

$$W_8(\eta)=-\frac{\nu}{2\zeta}+\frac{\sqrt{\phi}}{2\zeta}\left(-\tanh_\rho\left(\sqrt{\phi}\,\eta\right)\pm i\sqrt{mn}\,\mathrm{sech}_\rho\left(\sqrt{\phi}\,\eta\right)\right), \tag{23}$$

$$W_9(\eta)=-\frac{\nu}{2\zeta}+\frac{\sqrt{\phi}}{2\zeta}\left(-\coth_\rho\left(\sqrt{\phi}\,\eta\right)\pm\sqrt{mn}\,\mathrm{csch}_\rho\left(\sqrt{\phi}\,\eta\right)\right), \tag{24}$$

$$W_{10}(\eta)=-\frac{\nu}{2\zeta}-\frac{\sqrt{\phi}}{4\zeta}\left(\tanh_\rho\left(\frac{\sqrt{\phi}}{4}\eta\right)+\coth_\rho\left(\frac{\sqrt{\phi}}{4}\eta\right)\right). \tag{25}$$

(III) : If $\mu\zeta>0$ and $\nu=0$,

$$W_{11}(\eta)=\sqrt{\frac{\mu}{\zeta}}\tan_\rho\left(\sqrt{\mu\zeta}\,\eta\right), \tag{26}$$

$$W_{12}(\eta)=-\sqrt{\frac{\mu}{\zeta}}\cot_\rho\left(\sqrt{\mu\zeta}\,\eta\right), \tag{27}$$

$$W_{13}(\eta)=\sqrt{\frac{\mu}{\zeta}}\left(\tan_\rho\left(2\sqrt{\mu\zeta}\,\eta\right)\pm\sqrt{mn}\sec_\rho\left(2\sqrt{\mu\zeta}\,\eta\right)\right), \tag{28}$$

$$W_{14}(\eta)=\sqrt{\frac{\mu}{\zeta}}\left(-\cot_\rho\left(2\sqrt{\mu\zeta}\,\eta\right)\pm\sqrt{mn}\csc_\rho\left(2\sqrt{\mu\zeta}\,\eta\right)\right), \tag{29}$$

$$W_{15}(\eta)=\frac{1}{2}\sqrt{\frac{\mu}{\zeta}}\left(\tan_\rho\left(\frac{\sqrt{\mu\zeta}}{2}\eta\right)-\cot_\rho\left(\frac{\sqrt{\mu\zeta}}{2}\eta\right)\right). \tag{30}$$

(IV) : If $\mu\zeta<0$ and $\nu=0$,

$$W_{16}(\eta)=-\sqrt{-\frac{\mu}{\zeta}}\tanh_\rho\left(\sqrt{-\mu\zeta}\,\eta\right), \tag{31}$$

$$W_{17}(\eta)=-\sqrt{-\frac{\mu}{\zeta}}\coth_\rho\left(\sqrt{-\mu\zeta}\,\eta\right), \tag{32}$$

$$W_{18}(\eta)=\sqrt{-\frac{\mu}{\zeta}}\left(-\tanh_\rho\left(2\sqrt{-\mu\zeta}\,\eta\right)\pm i\sqrt{mn}\,\mathrm{sech}_\rho\left(2\sqrt{-\mu\zeta}\,\eta\right)\right), \tag{33}$$

$$W_{19}(\eta)=\sqrt{-\frac{\mu}{\zeta}}\left(-\coth_\rho\left(2\sqrt{-\mu\zeta}\,\eta\right)\pm\sqrt{mn}\,\mathrm{csch}_\rho\left(2\sqrt{-\mu\zeta}\,\eta\right)\right), \tag{34}$$

$$W_{20}(\eta) = -\frac{1}{2}\sqrt{-\frac{\mu}{\zeta}}\left(\tanh_\rho\left(\frac{\sqrt{-\mu\zeta}}{2}\eta\right) + \coth_\rho\left(\frac{\sqrt{-\mu\zeta}}{2}\eta\right)\right). \tag{35}$$

(V) : If $\nu = 0$ *and* $\mu = \zeta$,

$$W_{21}(\eta) = \tan_\rho(\mu\eta), \tag{36}$$

$$W_{22}(\eta) = -\cot(\mu\eta), \tag{37}$$

$$W_{23}(\eta) = \tan_\rho(2\mu\eta) \pm \sqrt{mn}\sec_\rho(2\mu\eta), \tag{38}$$

$$W_{24}(\eta) = -\cot_\rho(2\mu\eta) \pm \sqrt{mn}\csc_\rho(2\mu\eta), \tag{39}$$

$$W_{25}(\eta) = \frac{1}{2}\left(\tan_\rho\left(\frac{\mu}{2}\eta\right) - \cot_\rho\left(\frac{\mu}{2}\eta\right)\right). \tag{40}$$

(VI) : If $\nu = 0$ and $\zeta = -\mu$,

$$W_{26}(\eta) = -\tanh_\rho(\mu\eta), \tag{41}$$

$$W_{27}(\eta) = -\coth_\rho(\mu\eta), \tag{42}$$

$$W_{28}(\eta) = -\tanh_\rho(2\mu\eta) \pm i\sqrt{mn}\operatorname{sech}_\rho(2\mu\eta), \tag{43}$$

$$W_{29}(\eta) = -\coth_\rho(2\mu\eta) \pm \sqrt{mn}\operatorname{csch}_\rho(2\mu\eta), \tag{44}$$

$$W_{30}(\eta) = -\frac{1}{2}\left(\tanh_\rho\left(\frac{\mu}{2}\eta\right) + \coth_\rho\left(\frac{\mu}{2}\eta\right)\right). \tag{45}$$

(VII) : If $\nu^2 = 4\mu\zeta$,

$$W_{31}(\eta) = \frac{-2\mu(\nu\eta\ln\rho + 2)}{\nu^2\eta\ln\rho}. \tag{46}$$

$(VIII)$: If $\nu = \mathrm{p}, \mu = \mathrm{pq}, (q \neq 0)$ *and* $\zeta = 0$,

$$W_{32}(\eta) = \rho^{p\eta} - q. \tag{47}$$

(IX) : If $\nu = \zeta = 0$,

$$W_{33}(\eta) = \mu\eta\ln\rho. \tag{48}$$

(X) : If $\nu = \mu = 0$,

$$W_{34}(\eta) = \frac{-1}{\zeta\eta\ln\rho}. \tag{49}$$

(XI) : If $\mu = 0$ and $\nu \neq 0$,

$$W_{35}(\eta) = -\frac{m\nu}{\zeta(\cosh_\rho(\nu\eta) - \sinh_\rho(\nu\eta) + m)}, \tag{50}$$

$$W_{36}(\eta) = -\frac{\nu(\sinh_\rho(\nu\eta) + \cosh_\rho(\nu\eta))}{\zeta(\sinh_\rho(\nu\eta) + \cosh_\rho(\nu\eta) + n)}. \tag{51}$$

(XII) : If $\nu = p,\ \zeta = pq,\ (q \neq 0$ and $\mu = 0)$,

$$W_{37}(\eta) = -\frac{m\rho^{p\eta}}{m - qn\rho^{p\eta}}. \tag{52}$$

$$\sinh_\rho(\eta) = \frac{m\rho^\eta - n\rho^{-\eta}}{2},\ \cosh_\rho(\eta) = \frac{m\rho^\eta + n\rho^{-\eta}}{2},$$
$$\tanh_\rho(\eta) = \frac{m\rho^\eta - n\rho^{-\eta}}{m\rho^\eta + n\rho^{-\eta}},\ \coth_\rho(\eta) = \frac{m\rho^\eta + n\rho^{-\eta}}{m\rho^\eta - n\rho^{-\eta}},$$
$$\operatorname{sech}_\rho(\eta) = \frac{2}{m\rho^\eta + n\rho^{-\eta}},\ \operatorname{csch}_\rho(\eta) = \frac{2}{m\rho^\eta - n\rho^{-\eta}},$$

$$\sin_\rho(\eta) = \frac{m\rho^{i\eta} - n\rho^{-i\eta}}{2i},\ \cos_\rho(\eta) = \frac{m\rho^{i\eta} + n\rho^{-i\eta}}{2},$$
$$\tan_\rho(\eta) = -i\frac{m\rho^{i\eta} - n\rho^{-i\eta}}{m\rho^{i\eta} + n\rho^{-i\eta}},\ \cot_\rho(\eta) = i\frac{m\rho^{i\eta} + n\rho^{-i\eta}}{m\rho^{i\eta} - n\rho^{-i\eta}},$$
$$\sec_\rho(\eta) = \frac{2}{m\rho^{i\eta} + n\rho^{-i\eta}},\ \csc_\rho(\eta) = \frac{2i}{m\rho^{i\eta} - n\rho^{-i\eta}},$$

where m and n are arbitrary constants greater than zero and are called deformation parameters.

Applications:

Application for the $(1/G')$- *expansion method.*

Consider the MEW equation [51,52,55,59] with truncated M−Fractional derivative given as.

$$\frac{\partial^\beta u}{\partial t^\beta} + \theta\frac{\partial u^3}{\partial x} - \gamma\frac{\partial^2}{\partial x^2}\left(\frac{\partial^\beta u}{\partial t^\beta}\right) = 0. \tag{53}$$

Here $u = u(x, t)$ is the wave profile, while θ and γ are the parameters and$\beta \in (0, 1]$..

Let's assume the following traveling wave transformation:

$$u(x,\ t) = U(\eta),\ \ \eta = \frac{\Gamma(\beta+1)}{\alpha}(kx^\alpha - ct^\alpha), \tag{54}$$

where k and c are constants.

By using the Eq. (54) into the Eq. (53), we get the following ODE.

$$\gamma k^2 cU'' + \theta k(U^3)' - cU' = 0. \tag{55}$$

After one time integrate Eq. (55) w.r.tη, we get.

$$\gamma k^2 cU'' + \theta kU^3 - cU = 0. \tag{56}$$

By applying the homogenous balance technique between the terms U'' and U^3 into Eq. (56), we get $m = 1$. For $m = 1$, Eq. (5) reduces into:

$$U(\eta) = a_0 + a_1\left(\frac{1}{G'}\right), \tag{57}$$

where a_0 and a_1 are unknown parameters. By substituting Eq. (57) with Eq. (54) into Eq. (56) and summing up all the coefficients of same order of $(1/G')$, we get the algebraic equations involving a_0, a_1 and other parameters as follows:

$$\begin{aligned}
&\left(\frac{1}{G'}\right)^3 : a_1^3 k\theta + 2a_1 ck^2\gamma\mu^2 = 0,\\
&\left(\frac{1}{G'}\right)^2 : 3a_0 a_1^2 k\theta + 3a_1 ck^2\gamma\lambda\mu = 0,\\
&\left(\frac{1}{G'}\right)^1 : -a_1 c + 3a_0^2 a_1 k\theta + a_1 ck^2\gamma\lambda^2 = 0,\\
&\left(\frac{1}{G'}\right)^0 : -a_0 c + a_0^3 k\theta = 0.
\end{aligned} \tag{58}$$

Solving the system of algebraic equations in (58) with the help of software MATHEMATICA, we attain the following solutions:

$$a_0 = \pm\frac{\sqrt{c}}{\sqrt{k\theta}},\ a_1 = \pm\frac{2\sqrt{c}\mu}{\sqrt{k\theta}\lambda},\ \gamma = -\frac{2}{k^2\lambda^2}. \tag{59}$$

$$u(x,t) = \pm\frac{\sqrt{c}}{\sqrt{k\theta}}\left(1 + 2\mu\left(\frac{1}{-\mu + \lambda c_1[\cosh(\lambda\eta) - \sinh(\lambda\eta)]}\right)\right). \tag{60}$$

Application for the modified (G'/G^2)-expansion method.

By applying the homogenous balance technique into Eq. (56), we get $m=1$. For $m=1$, Eq. (9) reduces into:

$$U(\eta)=a_0+a_1\left(\frac{G'}{G^2}\right), \tag{61}$$

where a_0 and a_1 are unknown parameters. By using Eq. (61) with Eq. (4) into Eq. (56) and summing up all coefficients of same order of (G'/G^2), we get the set of algebraic equations involving a_0, a_1 and other parameters as follows:

$$\begin{aligned}
\left(\frac{G'}{G^2}\right)^3 &: a_1^3k\theta+2a_1ck^2\gamma\lambda_1^2=0,\\
\left(\frac{G'}{G^2}\right)^2 &: 3a_0a_1^2k\theta=0,\\
\left(\frac{G'}{G^2}\right)^1 &: -a_1c+3a_0^2a_1k\theta+2a_1ck^2\gamma\lambda_0\lambda_1=0,\\
\left(\frac{G'}{G^2}\right)^0 &: -a_0c+a_0^3k\theta=0.
\end{aligned} \tag{62}$$

Solving the system of algebraic equations in (62) with the help of software MATHEMATICA, we attain the following solutions:

$$a_0=0,\ a_1=\pm\frac{i\sqrt{c}\sqrt{\lambda_1}}{\sqrt{k\theta\lambda_0}},\ \gamma=\frac{1}{2k^2\lambda_0\lambda_1}. \tag{63}$$

Now we use Eqs. (63), (61) and (10) - (16) into Eq. (56) and set to the below cases.

if $\lambda_0\lambda_1<0$, then, we have hyperbolic traveling wave solution of Eq. (56).

$$u_1(x,t)=\mp\frac{i\sqrt{c}}{\sqrt{k\theta}}\left(1-\frac{\lambda_1}{2}\left[\frac{c_1\sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+c_2\cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{c_1\cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+c_2\sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right). \tag{64}$$

If $\lambda_0\lambda_1>0$, we have trigonometric traveling wave solution of Eq. (56).

$$u_2(x,t)=\pm\frac{i\sqrt{c}}{\sqrt{k\theta}}\left(\left[\frac{c_1\cos\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+c_2\sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{c_1\sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)-c_2\cos\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right). \tag{65}$$

Application for the new extended direct algebraic method.

By applyingIthe homogenous balance technique into Eq. (56), we get $m=1$. For $m=1$, Eq. (14) reduces into:

$$U(\eta)=a_0+a_1W(\eta), \tag{66}$$

where $W(\eta)$ satisfies the Eq. (15).

After plugging Eq. (66) into Eq. (56), we get a structure of the respective algebraic equations and coefficients of different powers of $W(\eta)$ are equalized.

$$\begin{aligned}
W^3(\eta) &: a_1^3k\theta+2a_1ck^2\ln^2\gamma\zeta^2\rho^2=0,\\
W^2(\eta) &: 3a_0a_1^2k\theta+3a_1ck^2\ln^2\gamma\zeta^2\nu\rho^2=0,\\
W^1(\eta) &: -a_1c+3a_0^2a_1k\theta+2a_1ck^2\ln^2\gamma\zeta\mu\rho^2+a_1ck^2\ln^2\gamma\nu^2\rho^2=0,\\
W^0(\eta) &: -a_0c+a_0^3k\theta+a_1ck^2\ln^2\gamma\mu\nu\rho^2=0.
\end{aligned} \tag{67}$$

Employing computational program to solve the above algebraic equations, the following set of solution is obtained:

$$a_0=\pm\frac{2\sqrt{c}\zeta\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}\nu},\ a_1=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}},\ \gamma=-\frac{2}{k^2\ln^2\rho^2\Phi}. \tag{68}$$

$\Phi=\nu^2-4\mu\zeta.$

Set 1. When $\Phi<0$ along with $\zeta\neq0$, then.

After plugging the values of a_0 and a_1 via Eq. (68) into Eq. (66), which represents the regarding solutions of Eq. (67):

$$u_1(x,t)=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp1\pm\frac{\sqrt{-\Phi}}{\nu}\tan_\rho\left(\frac{\sqrt{-\Phi}}{2}\eta\right)\right), \tag{69}$$

hence the corresponding solutions are extracted, functioning in much the same line.

$$u_2(x,t)=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp1\mp\frac{\sqrt{-\Phi}}{\nu}\cot_\rho\left(\frac{\sqrt{-\Phi}}{2}\eta\right)\right), \tag{70}$$

$$u_3(x,t)=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp1\pm\frac{\sqrt{-\Phi}}{\nu}\left(\tan_\rho(\sqrt{-\Phi}\,\eta)\pm\sqrt{mn}\,sec_\rho(\sqrt{-\Phi}\,\eta)\right)\right), \tag{71}$$

$$u_4(x,t)=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp1\pm\frac{\sqrt{-\Phi}}{\nu}\left(\cot_\rho(\sqrt{-\Phi}\,\eta)\pm\sqrt{mn}\,cosec_\rho(\sqrt{-\Phi}\,\eta)\right)\right), \tag{72}$$

$$u_5(x,t)=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp1\pm\frac{\sqrt{-\Phi}}{2\nu}\left(\tan_\rho\left(\frac{\sqrt{-\Phi}}{4}\eta\right)-\cot_\rho\left(\frac{\sqrt{-\Phi}}{4}\eta\right)\right)\right). \tag{73}$$

Set 2. When $\Phi>0$ along with $\zeta\neq0$, then.

$$u_6(x,t)=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp1\mp\frac{\sqrt{\Phi}}{\nu}\tanh_\rho\left(\frac{\sqrt{\Phi}}{2}\eta\right)\right), \tag{74}$$

$$u_7(x,t)=\pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp1\mp\frac{\sqrt{\Phi}}{\nu}\coth_\rho\left(\frac{\sqrt{\Phi}}{2}\eta\right)\right), \tag{75}$$

$$u_8(x,t) = \pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp 1\pm\frac{\sqrt{\Phi}}{\nu}\left(-\tanh_\rho\left(\sqrt{\Phi}\eta\right)\pm i\sqrt{mn}\,\mathrm{sech}_\rho\left(\sqrt{\Phi}\eta\right)\right)\right), \tag{76}$$

$$u_9(x,t) = \pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp 1\pm\frac{\sqrt{\Phi}}{\nu}\left(-\coth_\rho\left(\sqrt{\Phi}\eta\right)\pm \sqrt{mn}\,cosech_\rho\left(\sqrt{\Phi}\eta\right)\right)\right), \tag{77}$$

$$u_{10}(x,t) = \pm\frac{\sqrt{c}\sqrt{1+\frac{4\zeta\mu}{\Phi}+\frac{2\nu^2}{\Phi}}}{\sqrt{3k\theta}}\left(1\mp 1\pm\frac{\sqrt{\Phi}}{2\nu}\left(\tanh_\rho\left(\frac{\sqrt{\Phi}}{4}\eta\right)\pm\coth_\rho\left(\frac{\sqrt{\Phi}}{4}\eta\right)\right)\right). \tag{78}$$

Set 3. When $\nu = p, \mu = pq, (q = 0)$ and $\zeta = 0$, then.

$$u_{11}(x,t) = 3(\rho^{p\eta} - q). \tag{79}$$

Set 4. When $\mu = 0$ and $\nu \neq 0$, then.

$$u_{12}(x,t) = -3\left(\frac{m\nu}{\zeta(\cosh_\rho(\nu\eta) - \sinh_\rho(\nu\eta) + m)}\right), \tag{80}$$

$$u_{13}(x,t) = -3\left(\frac{\nu(\sinh_\rho(\nu\eta) + \cosh_\rho(\nu\eta))}{\zeta(sinh_\rho(\nu\eta) + \cosh_\rho(\nu\eta) + n)}\right). \tag{81}$$

Set 8. When $\nu = p,\ \zeta = pq,\ (q \neq 0$ and $\mu = 0)$, then.

$$u_{14}(x,t) = -3\left(\frac{m\rho^{p\eta}}{m - qn\rho^{p\eta}}\right). \tag{82}$$

Dynamical behaviour

This section aims to study the dynamical behavior of the fractional MEW equation and the influence of the system's parameters on certain type of the solutions.

Equation (56) can be expressed as a 2D-dynamical system in the form.

$$U' = y, y' = -AU + BU^3, \tag{83}$$

where $A = \frac{-1}{\gamma k^2}, B = \frac{-\theta}{c\gamma k}$ are new constants that are introduced for simplicity. The dynamical system (83) is a conservative Hamiltonian system with one degree of freedom describing physically the one-dimension motion of a unit mass particle. Its Hamilton function admits the form.

$$H = \frac{1}{2}y^2 + \frac{A}{2}U^2 - \frac{B}{4}U^4. \tag{84}$$

It is obvious the Hamiltonian (84) does not rely explicitly on the independent variable η which plays the role of the time in Hamiltonian mechanics. Hence, it has a conserved quantity in the form.

$$\frac{1}{2}y^2 + \frac{A}{2}U^2 - \frac{B}{4}U^4 = h, \tag{85}$$

where h is an arbitrary parameter. Employing the first equation in (83) and the conserved quantity (85) and separating the variables, we obtain the next one-differential form.

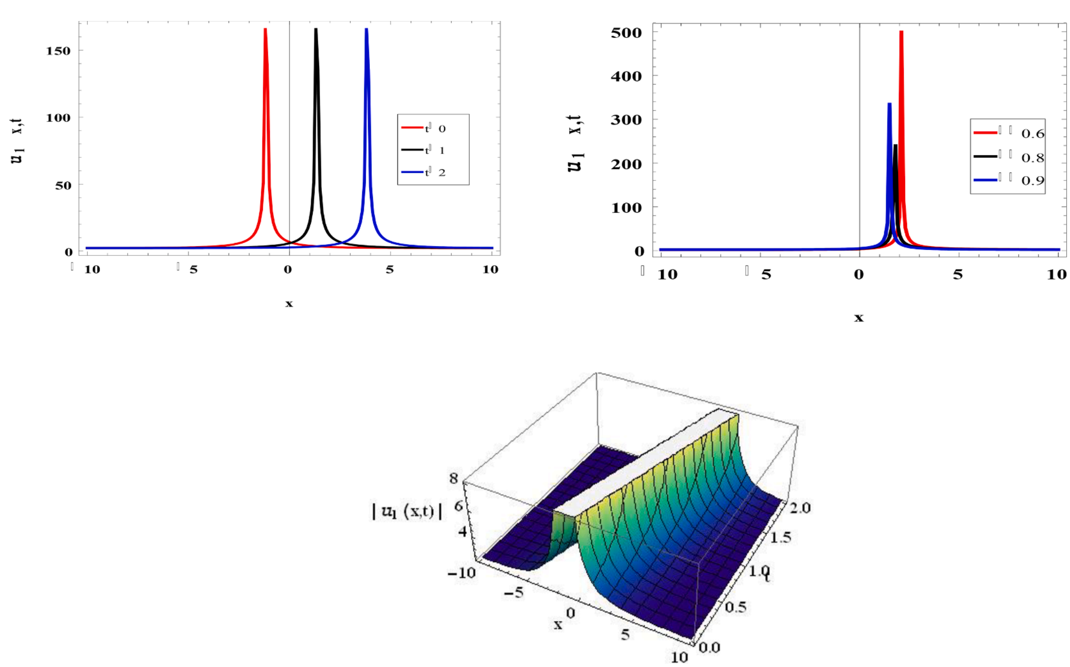


Fig. 1. 2D and 3D graphics of hyperbolic traveling wave solution for Eq. (60) at..$\{\theta = 0.5, \lambda_0 = 0.5, \lambda_1 = 1, \beta = 4, C_1 = 1, c = 0.5, k = 0.2\}$

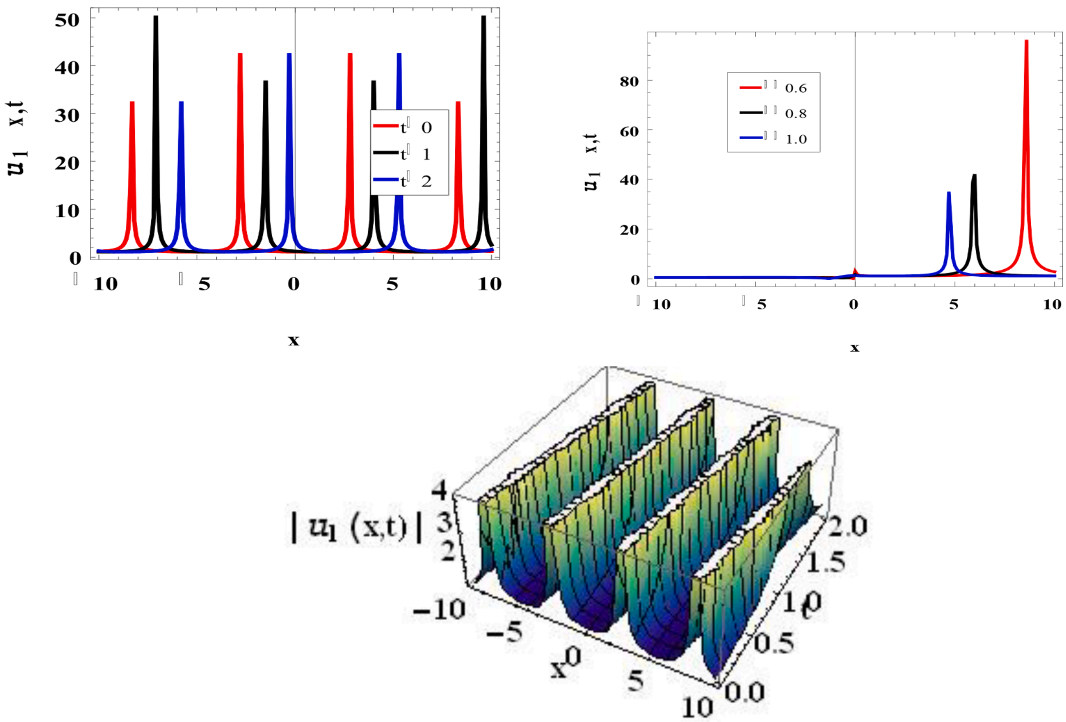


Fig. 2. 2D and 3D graphics of hyperbolic periodic traveling wave solution for Eq. (64) at..$\{k = 0.8, \theta = 1, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_1 = 0, C_2 = 1, c = 1\}$

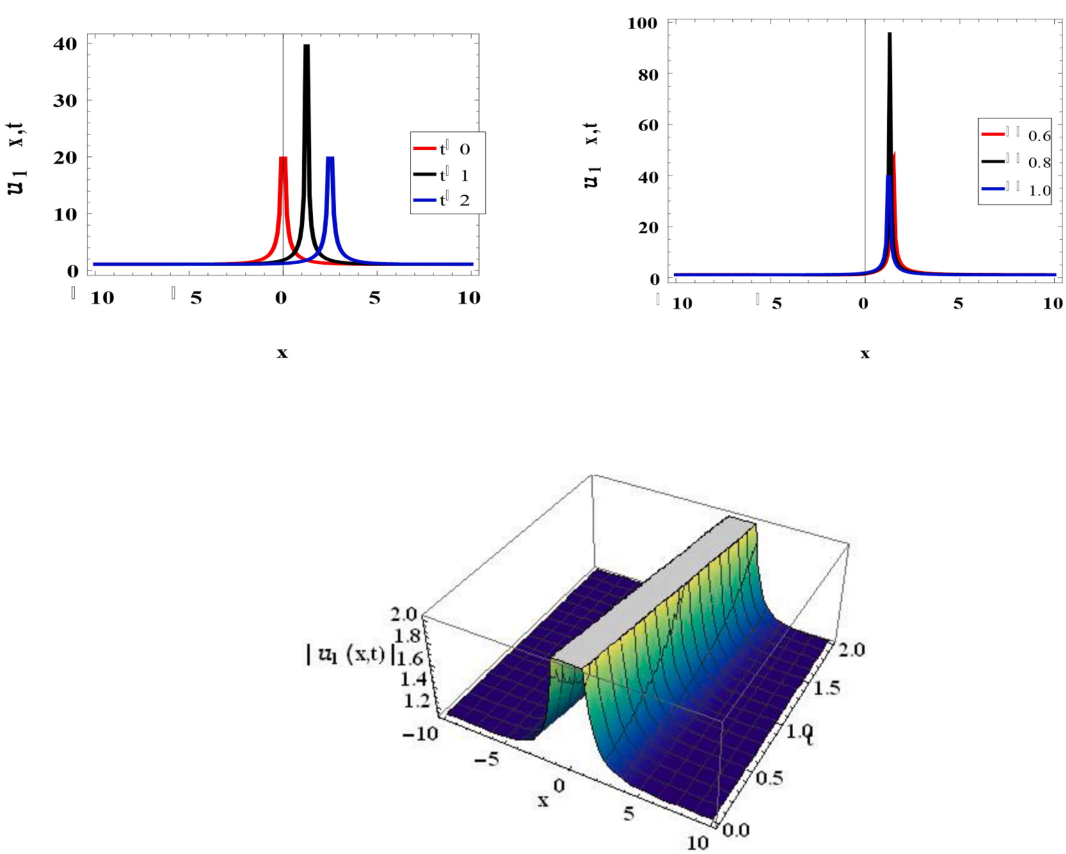


Fig. 3. 2D and 3D graphics of trigonometric traveling wave solution for Eq. (65) at..$\{k = 0.8, \theta = 1, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_1 = 1, C_2 = 0, c = 1\}$

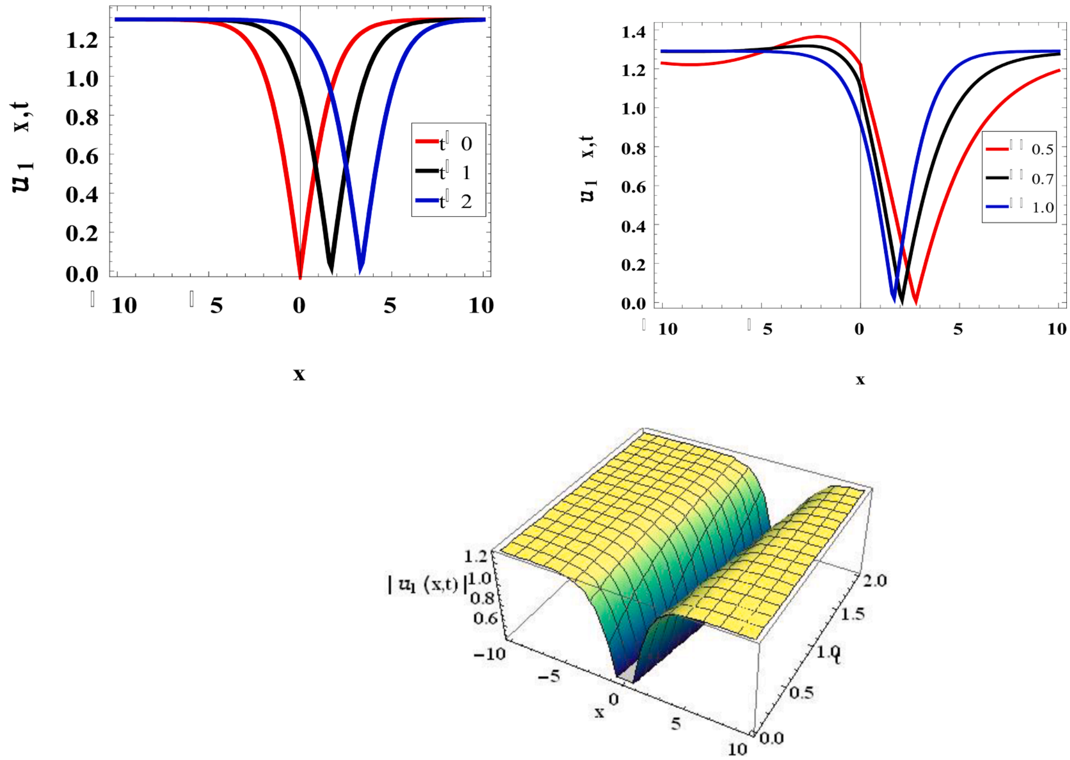


Fig. 4. 2D and 3D graphics of trigonometric traveling wave solution for Eq. (69) at..$\{k = 0.6, \theta = 1, \mu_0 = 0.8, \zeta = 1, \beta = 2, c = 1, \nu = 2\}$

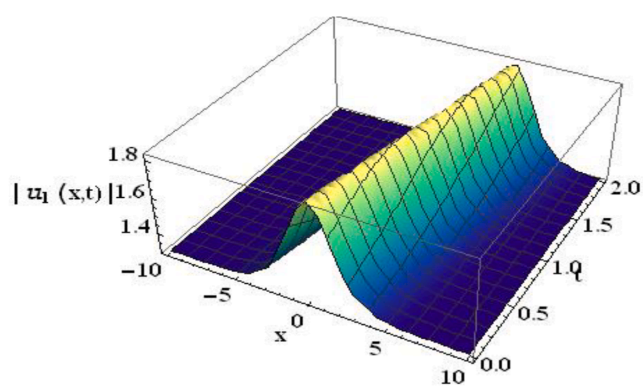


Fig. 5. 3D graph of trigonometric traveling wave solution for Eq. (71) at..$\{k = 0.6, \theta = 1, \mu_0 = 0.8, \zeta = 1, \beta = 2, c = 1, \nu = 2, m = 2, n = 1\}$

$$\frac{dU}{\sqrt{Q_4(U)}} = \frac{1}{\sqrt{2}}d\eta, \tag{86}$$

where.

$$Q_4(U) = B\left[U^4 - \frac{2A}{B}U^2 + \frac{4h}{B}\right]. \tag{87}$$

To integrate both sides of Eq. (86), the range of the parameter A, B, h are required. One of the most significant methods to find this range is the bifurcation analysis. This method acquires their significant from it gives the required range of the parameters and at the same time determines the type of the solution before constructing them. For instance, the existence of periodic, heteroclinic, and homoclinic orbits for the system (83) refers to the presence of periodic, kink(anti-kink), and solitary wave solution. This method was applied successfully in various works, see, e.g., [60–64].

Now, we are going to apply the qualitative theory for planar dynamical system for the system (83) [65]. The system (83) has a unique equilibrium point $E_1 = (0,0)$ if $AB < 0$ while it has three equilibrium points $E_1 = (0,0), E_{2,3} = (\pm\sqrt{\frac{A}{B}}, 0)$. The natural of these equilibrium points can be determined by using the eigenvalue of the Jacobi matric corresponding the system (83). It has the form $\lambda_{1,2} = \pm\sqrt{3BU^2 - A}$. The values these eigenvalues at the equilibrium points are Figs. 1–9.

$$\lambda_{1,2}(E_1) = \pm\sqrt{-A}, \lambda_{1,2}(E_{2,3}) = \pm\sqrt{A}. \tag{88}$$

Thus, we have the following two cases:

- If $AB < 0$, the system (83) has one equilibrium point E_1 which is a saddle if $A < 0, B > 0$ and it is a center point if $A > 0, B < 0$. Fig. 12 (a) and (b) outline this case.
- If $AB > 0$, the system (83) has three equilibrium points $E_{1,2,3}$. If $A > 0, B > 0$, then E_1 is a center point and $E_{2,3}$ are saddle points while if $A < 0, B < 0$, then E_1 is a saddle point and $E_{2,3}$ are center points. Fig. 10 (c) and (d) outline this case.

The value of the parameter h at the equilibrium points are:

$$h_1(E_1) = 0, h_2 = \frac{A^2}{4B}. \tag{89}$$

As we know, any phase orbit is an energy level curve, which is described as.

$$C_h = \{(U, y) \in \mathbb{R}^2 : H = h\}. \tag{90}$$

The phase portrait outlined by Fig. 10 can be describe in the following item.

- If $A < 0, B > 0$, all the phase orbits are unbounded for all values of the parameter h. Hence, the integration of equation (86) along any of these orbits give unbounded wave solutions.
- If $A > 0, B < 0$, the system (83) as a unique family of bounded periodic phase orbit. Consequently, the integration of both sides of equation (86) implies to bounded wave solution which is periodic.

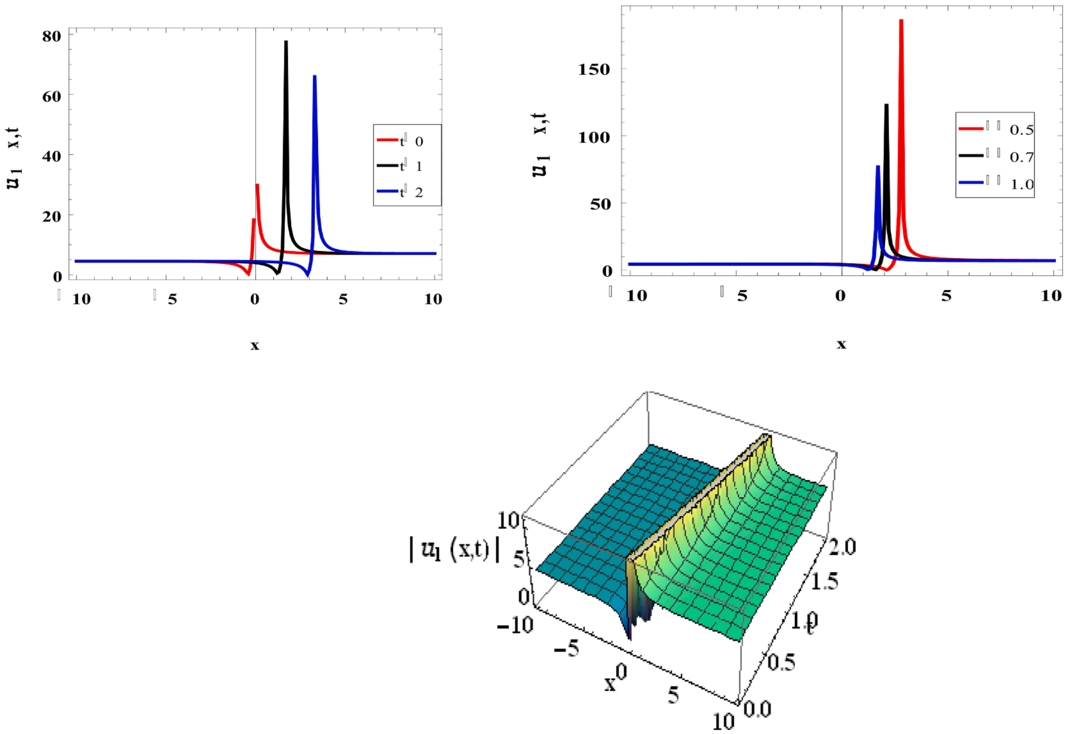


Fig. 6. 2D and 3D graphics of hyperbolic traveling wave solution for Eq. (75) at..$\{k = 0.6, \theta = 1, \mu_0 = 0.8, \zeta = 1, \beta = 2, c = 1, \nu = 2\}$

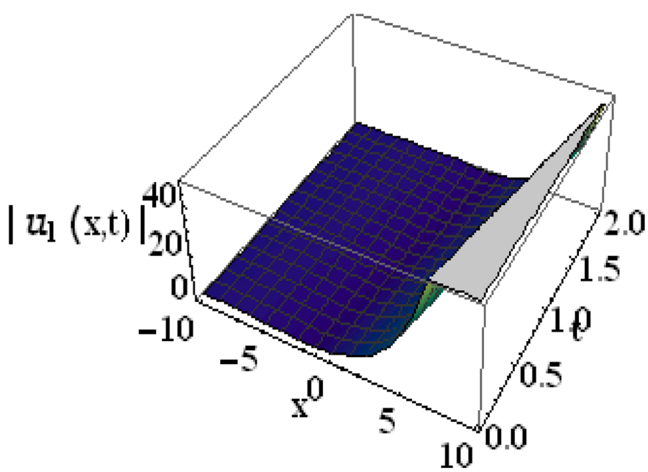


Fig. 7. 3D graph of exponential traveling wave solution for Eq. (79) at..$\{k = 0.6, p = 0.5, q = 0.5, \beta = 2, c = 1, \rho = 2\}$

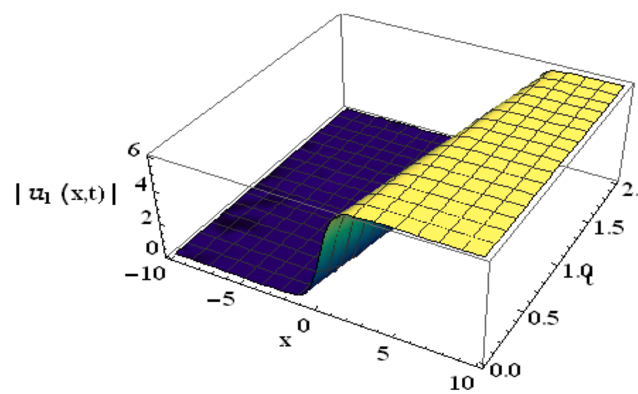


Fig. 8. 3D graph of traveling wave solution for Eq. (81) at..$\{k = 0.6, \zeta = 1, \beta = 2, c = 1, \nu = 2, n = 2\}$

- If $A > 0, B > 0$, the system (83) has different families of orbits in which some is bounded, and the other is unbounded. For $h > h_2$, there is a family of unbounded orbit in green $\{C_h : h > h_2\}$. For $h = h_2$, there is a heteroclinic orbit in red connecting the two saddle points $E_{2,3}$ and this type of orbit refers to the existence of kink (or anti-kink) wave solution. For $h \in]0, h_2[$, there are three types of phase orbits in blue. Two of them are unbounded and they arise outside the heteroclinic orbit C_{h_2} while the other is periodic, and it appears inside the heteroclinic orbit. For $h = 0$ and $h < 0$, there are unbounded phase orbits in black and brown respectively.
- If $A < -1, B < -1$, the system (83) has different families of bounded orbits. For $h \in]h_2, 0[$, there are two families of periodic orbits in green which are placed inside the homoclinic orbit $C_{h=0}$. The existence of homoclinic orbit refers to the existence of solitary wave solution. For $h > 0$, there is a family of super periodic orbit in blue which implies to the existence of super periodic wave solutions.

In the following subsection, we restrict ourselves to construct only bounded wave solutions, which are related with the phase bounded orbits. Moreover, we study the influence of the parameters on the types of the solutions.

Wave solutions

Solitary wave solution

This type of solution corresponding to the homoclinic orbit, which exists if $A < 0, B < 0, h = 0$. The solitary solution has the form.

$$U = \sqrt{\frac{2A}{B}} \operatorname{sech} \sqrt{-A}(\eta - \eta_0). \tag{91}$$

where η_0 is an integration constant.

Kink (anti-kink) solution

This kind of the solution associated with the heteroclinic orbit which arise if $A > 0, B > 0, h = h_2$ as outlined by Fig. 10(c) in red which

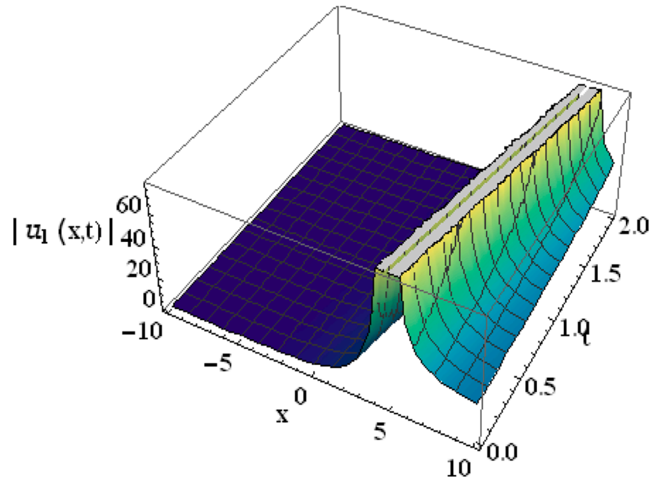


Fig. 9. 3D graph for traveling wave solution for Eq. (82) at..$\{k = 0.6, p = 0.5, q = 0.3, \beta = 2, c = 1, \rho = 2, m = 2, n = 1\}$

connected the two saddle points. Thus, we have a solution in the form.

$$U = \sqrt{\frac{A}{B}}\tanh\sqrt{\frac{A}{2}}(\eta - \eta_0), \tag{92}$$

where η_0 is the integration constant.

Periodic solution

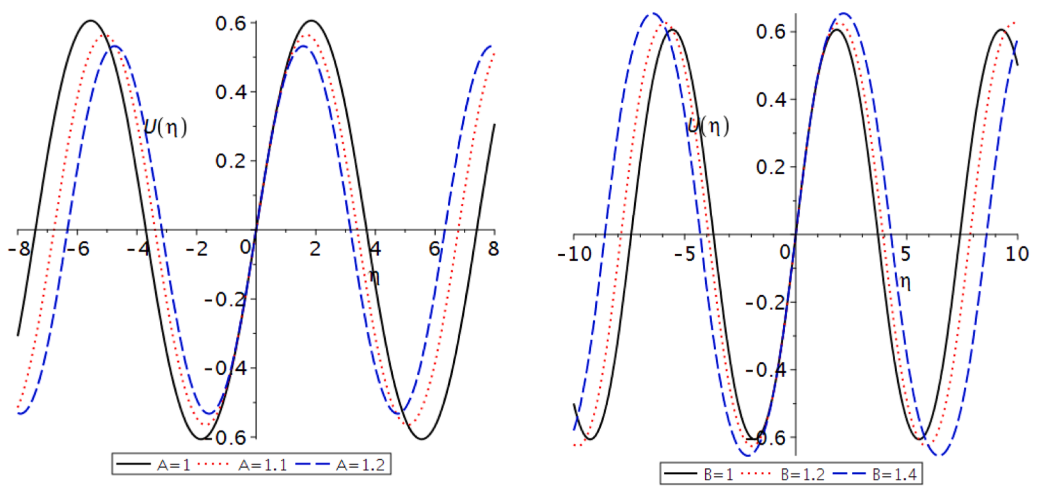
As outlined by the phase portrait in Fig. 10, there are several periodic orbits, which are related to the existence of the periodic solutions. Let us calculate them individually:

- For $A > 0, B < 0, h > 0$, there is one family of periodic orbits around the center point $E_1(0,0)$ in blue as illustrated in blue by Fig. 12(b). Integrating equation (86) along one of these orbits, we obtain

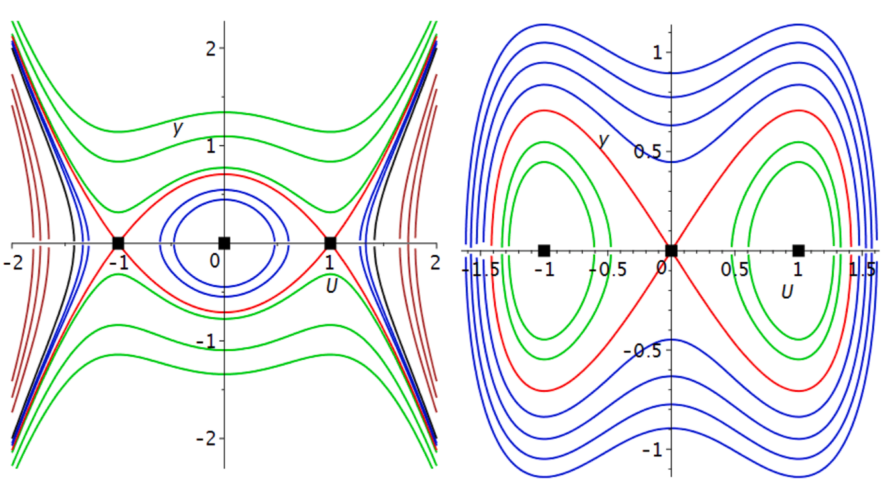
$$U = u_1 \mathrm{cn}\left(\sqrt{-\frac{B}{2}(u_1^2 + u_2^2)}\,(\eta - \eta_0), \frac{u_1}{\sqrt{u_1^2 + u_2^2}}\right) \tag{93}$$

where u_1^2, u_2^2 are the roots of the quadratic polynomial $X^2 - \frac{2A}{B}X + \frac{4h}{B}$.

- For $A > 0, B > 0, h \in]0, h_2[$, there is a family of periodic orbits in blue around the center point $E_1 = (0,0)$ that lie inside the hetroclinic

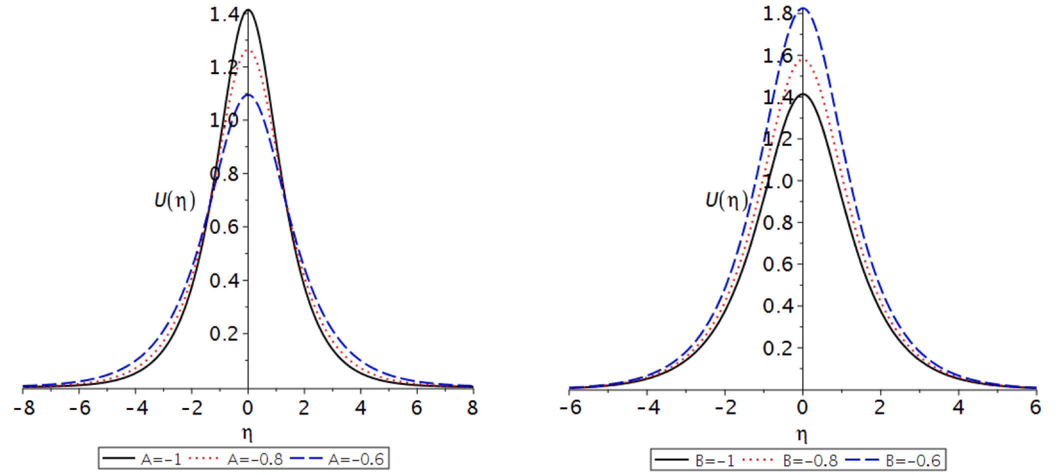


(a) The effect of parameter A with fixing $B = 1$ (b) The effect of parameter B with fixing $A = 1$



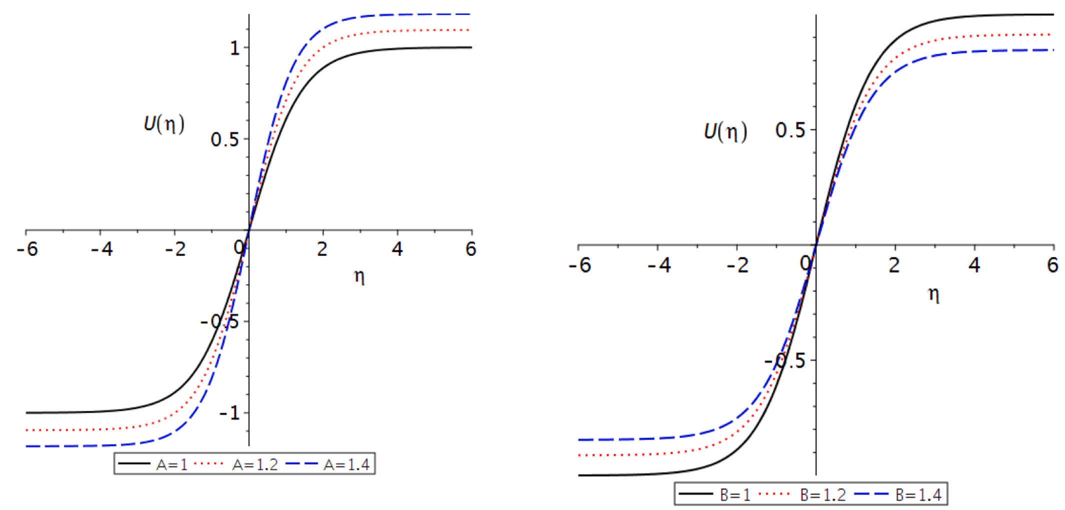
(c) $A = B = 1$ **(d)** $A = B = -1$

Fig. 10. Phase portrait for the dynamical system (83). The black solid boxes indicate the equilibrium points.



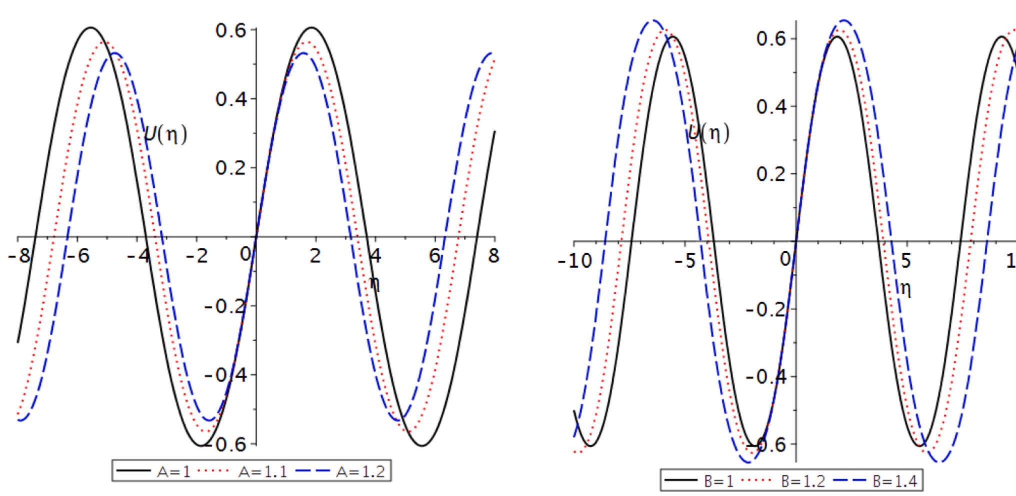
(a) The effect of parameter A with fixing $B=-1$ (b) The effect of parameter B with fixing $A=-1$

Fig. 11. The influence of parameters on the solitary solution (91).



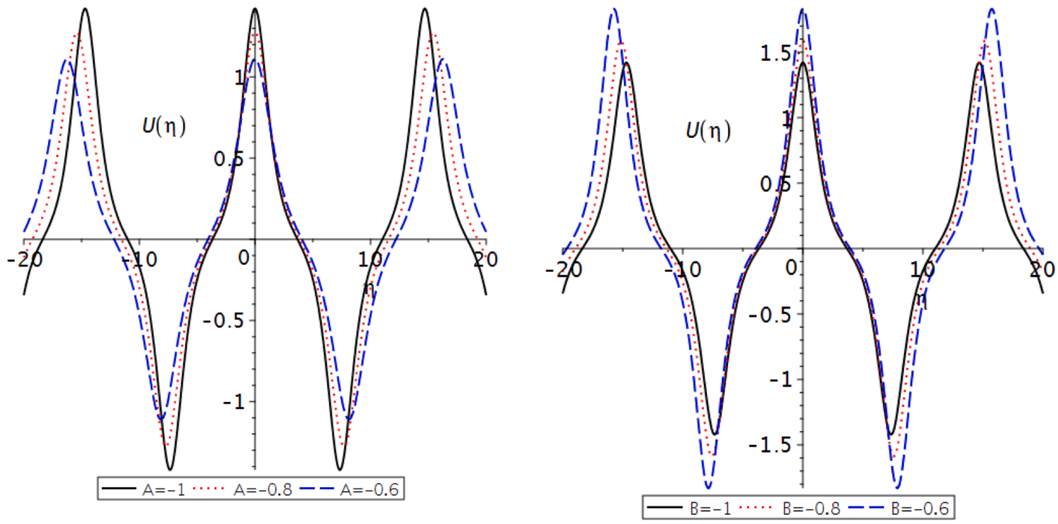
(a) The effect of parameter A with fixing $B=1$ (b) The effect of parameter B with fixing $A=1$

Fig. 12. The influence of parameters on the kink solution (92).



(a) The effect of parameter A with fixing $B=1$ (b) The effect of parameter B with fixing $A=1$

Fig. 13. The influence of parameters on the kink solution (94).



(a) The effect of parameter A with fixing $B = -1$ (b) The effect of parameter B with fixing $A = -1$

Fig. 14. The influence of parameters on the kink solution (94).

orbit in red, see Fig. 10 (c). Integrating both sides of Eq. (86) along one orbit of this family, we get

$$U = u_1 \mathrm{sn}\left(u_2\sqrt{\frac{B}{2}}(\eta-\eta_0), \frac{u_1}{u_2}\right), \tag{94}$$

where u_1^2, u_2^2 are the roots of the quadratic polynomial $X^2 - \frac{2A}{B}X + \frac{4h}{B}$ with $0 < u_1 < u_2$.

- For $A < 0, B < 0, h \in]h_2, 0[$, there is two families of orbits in green about the two centers point $E_{2,3}$ as clarified in green by Fig. 10 (d). Integrating Eq. (86) along one of these orbits, we have

$$U = u_2 \mathrm{dn}\left(\sqrt{-\frac{B}{2}}(\eta-\eta_0), \sqrt{1-\frac{u_1^2}{u_2^2}}\right), \tag{95}$$

where u_1^2, u_2^2 are the roots of the quadratic polynomial $X^2 - \frac{2A}{B}X + \frac{4h}{B}$ with $0 < u_1 < u_2$.

Super periodic solution

For $A < 0, B < 0, h > 0$, there is a family of super periodic orbits around the two equilibrium points $E_{2,3}$ as illustrated by Fig. 10 (d). Integrating along one of these orbits, we obtain a solutions.

$$U = u_1 \mathrm{cn}\left(\sqrt{-\frac{B}{2}(u_1^2+u_2^2)}\,(\eta-\eta_0), \frac{u_1}{\sqrt{u_1^2+u_2^2}}\right), \tag{96}$$

where u_1^2, u_2^2 are the roots of the quadratic polynomial $X^2 - \frac{2A}{B}X + \frac{4h}{B}$ with $0 < u_1 < u_2$.

Parameters affecting solutions

This section aims to study numerically the influence of the parameters θ, γ for fixed values of k, c on the obtained solutions in the previous subsection.

Fig. 11 describes the effect of the two parameters A, B on the solitary wave solution (91). For fixing B, Fig. 11(a) outlines the amplitude of wave solution (91) decrease while the width increase with increasing A. Keeping A fixed, Fig. 11(b) clarifies both amplitude and width of the wave solution (91) increase when B increases.

Fig. 12 illustrates the effect of the parameter A, B on the kink wave solution (92). For fixing B, the amplitude of the kink solution (92) increases with increasing A. But for making A fixed, the amplitude decreases with increasing A.

Fig. 13 outlines the impact of the parameters A, B on the periodic wave solution (94). Keeping B fixed, both the amplitude and the width of the wave solution (94) decrease with increasing A as outlined by Fig. 13 (a). But for keeping A fixed, both the amplitude and the width of the wave solution increases with increasing the growth of the parameter B.

Fig. 14 illustrates the impact of the parameters A, B on the super wave solution (96). Fig. 14 (a) shows both the amplitude and the width of the super wave solution (96) decrease with keeping B fixed. Whereas Fig. 14(b) outlines the amplitude and the width of the super wave solution (96) increase with fixing A.

Results and discussions

To show the dynamics and behavior of our obtained solutions, various exat traveling wave solutions in Eqs. (60), (64), (65), (69), (71), (75), (79), (81) and (82) are graphically represented and compared in both 3D and 2D plots in Figs. 1–9 for various parameters' values. A 3D plot highlights the amount of variation over a while or compares multiple wave items. The 2D line plots are used to represents very high and low frequency and amplitude. The plots are constructed with unique values of $\alpha \in (0, 1]$ for different values of free parameters. The plots denote many natures, such as the trigonometric, hyperbolic and rational wave solutions and other forms of the solution generated by the correct physical description by choosing different free parameters. We can observe the plotted graphs Figs. 1-9 that the wave's frequency and amplitude change with the change of fractional and time parameters. In the concept of mathematical physics, a soliton or solitary wave is defined as a self-reinforcing wave packet that upholds its shape. At the same time, it propagates at a constant amplitude and velocity. Soliton is the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. These soliton type solutions key physical structures are displayed in 4, namely their trajectories, solitary wave solutions, periodic wave solutions, king and antiking wave solutions and most prominent supernonlinear periodic wave solutions.

Conclusions

In this paper, we applied the $(1/G')$- expansion method, modified

(G'/G^2)- expansion method and the new extended direct algebraic methods in a satisfactory way to find the exact solutions of the M−fractional MEW equation. Various obtained solutions are in the form of hyperbolic, trigonometric and rational forms are entirely different forms which have been reported in previously published studies [49–56]. We have also depicted some of the obtained solutions graphically and concluded that the obtained results are accurate, efficient and versatile in mathematical physics to solve other nonlinear fractional differential equations. Further, by using bifurcation theory, we successfully studied the bifurcation behavior of nonlinear fractional MEW equation. It was seen that nonlinear fractional MEW equation underpins nonlinear solitary wave, periodic wave, king and antiking wave and most prominent supernonlinear periodic waves.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgement

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References

[1] Khalil R, Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. J Comput Applied Math 2014;264:65–70.
[2] Hosseini K, Mirzazadeh M, Ilie M, Gómez-Aguilar JF. Biswas-Arshed equation with the beta time derivative : Optical solitons and other solutions. Optik 2020;217:164801.
[3] Vanterler da C, Sousa J, Capelas de Oliveira E. A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties. Inter J Analysis Appli 2018;16(1):83–96.
[4] Ghanbari B, Gómez-Aguilar JF. New exact optical soliton solutions for nonlinear Schrödinger equation with second-order spatio-temporal dispersion involving M-derivative. Mod Phys Lett B 2019;33(20):1–9.
[5] Kaur L, Wazwaz AM. Painleve analysis and invariant solutions of generalized fifth-order nonlinear integrable equation. Nonlinear Dyn 2018;94:2469–77.
[6] Kaur L, Wazwaz AM. Einstein's vacuum field equation: painleve analysis and Lie Symmetries. Waves Random Complex Medium 2021;31(2):199–206.
[7] Wazwaz A-M, Kaur L. Complex simplified Hirota's forms and Lie symmetry analysis for multiple real and complex soliton solutions of the modified KdV-Sine-Gordon equation. Nonlinear Dyn 2019;95(3):2209–15.
[8] Kaur L, Wazwaz AM. Dynamical analysis of soliton solutions for space-time fractional calogero-degasperis and sharma-tasso-olver equations. Rom Rep Phys 2022:1–17. http://www.infim.ro/rrp.
[9] Hosseini K, Mirzazadeh M, Ilie M, Radmehr S. Dynamics of optical solitons in the perturbed gerdjikov-ivanov equation. Optik 2020;206:164350.
[10] Hosseini K, Kaur L, Mirzazadeh M, Baskonus HM. Soliton solutions of the (2+1)-dimensional Heisenberg ferromagnetic spin chain model with the beta time derivative. Opt Quant Electron 2021;53:125.
[11] Jaradat I, Alquran M, Qureshi S, Sulaiman TA, Yusuf A. Convex-rogue, half-kink, cusp-soliton and other bidirectional wave-solutions to the generalized Pochhammer-Chree equation. Phys Scr 2022;97(5):055203.
[12] Al-Smadi M, Abu Arqub O, Hadid S. Approximate solutions of nonlinear fractional Kundu-Eckhaus and coupled fractional massive Thirring equations emerging in quantum field theory using conformable residual power series method. Phys Scr 2020;95(10):105205.
[13] Al- SM. Fractional residual series for conformable time- fractional Sawada-Kotera-lto, Lax and Kaup-Kupershmidt equations of seventh order. Mathematical Methods in the Applied Sciences 2021. https://doi.org/10.1002/mma.7507.
[14] Bulut H, Aksan EN, Kayhan M, Sulaıman TA. New solitary wave structures to the (3 + 1) dimensional Kadomtsev-Petviashvili and Schrödinger equation. J Ocean Eng Sci 2019;9:373–8.
[15] Sulaiman TA. Three-component coupled nonlinear Schrödinger equation: optical soliton and modulation instability analysis. Phys Scr 2020;95(6):065201.
[16] Sulaiman TA, Bulut H, Baskonus HM, Cattani C, Atangana A, Bulut H, et al. Construction of various soliton solutions via the simplified extended sinh-Gordon equation expansion method. ITM Web of Conferences 2018;22:01062.
[17] Bulut H, Sulaıman TA, Baskonus HM. On the solitary wave solutions to the longitudinal wave equation in MEE circular rod. Opt Quant Electron 2018;50(2):1–10.
[18] Bulut H, Sulaıman TA, Baskonus HM, Akturk T. Complex acoustic gravity wave behaviors to some mathematical models arising in fluid dynamics and nonlinear dispersive media. Opt Quant Electron 2018;50(1):1–13.
[19] Cinar M, Onder I, Secer A, Bayram M, Yusuf A, Sulaiman TA. A comparison of analytical solutions of nonlinear complex generalized Zakharov dynamical system for various definitions of the differential operator. Electronic Research Archive 2022;30(1):335–61.
[20] Sulaıman TA, Yusuf A, Atangana A. New lump, lump-kink, breather waves and other interaction solutions to the (3+1)-dimensional soliton equation. Commun Theor Phys 2020;72(7):085004.
[21] Ozdemir N, Esen H, Secer A, Bayram M, Yusuf A, Sulaiman TA. Optical solitons and other solutions to the Hirota-Maccari system with conformable, M-truncated and beta derivatives. Mod Phys Lett B 2022;36(11).
[22] Zhang Z. New exact traveling wave solutions for the nonlinear klein-gordon equation. Turkish J Phys 2008;32:235–40.
[23] Zhang Z. Exact traveling wave solutions of the perturbed Klein-Gordon equation with quadratic nonlinearity in (1+1)-dimension, Part I-without local inductance and dissipation effect. Turk J Phys 2012;37(2):259–67.
[24] Zhang ZY, Liu ZH, Miao XJ, Chen YZ. New exact solutions to the perturbed nonlinear Schrodingers equation with Kerr law nonlinearity. Appl Math Comput 2010;216:3064–72.
[25] Zhang ZY, Li YX, Liu ZH. New exact solutions to the perturbed nonlinear Schrodingers equation with Kerr law nonlinearity via modified trigonometric function series method. Commun Nonlinear Sci Numerical Simul 2011;16(8):3097–106.
[26] Zhang ZY, Liu ZH, Miao XJ, Chen YZ. Qualitative analysis and traveling wave solutions for the perturbed nonlinear Schrodingers equation with Kerr law nonlinearity. Physics Letter A 2011;375:1275–80.
[27] Miao X, Zhang Z. The modified (G'/G)-expansion method and traveling wave solutions of nonlinear the perturbed nonlinear Schrodingers equation with Kerr law nonlinearity. Commu Nonli Sci Numerical Simu 2011;16(11):4259–67.
[28] Zhang Z. The extended (G'/G)-expansion method and travelling wave solutions for the perturbed nonlinear Schrodingers equation with Kerr law nonlinearity. Pramana 2014;82(6):1011–29.
[29] Zhang ZY, Zhong J, Dos SS, Liu J, Peng D, Gao T. A new method to construct traveling wave solutions for the Klein-Gordon Zakharov equations. Rom Journ Phys 2013;58(7–8):766–77.
[30] Siddique I, Rizvi STR, Batool F. New Exact Traveling Wave Solutions of Nonlinear Evolution Equations. Int J Nonlinear Sci 2010;9(1):12–8.
[31] Yun ZZ, Yang GX, Min YD, Hui ZY, Ping LX. A note on exact traveling wave solutions of the perturbed nonlinear Schrodingers equation with Kerr law nonlinearity. Commun Theor Phys 2012;57:764–70.
[32] Zhang Z, Wu J. Generalized (G'/ G)-expansion method and exact traveling wave solutions of the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity in optical fiber materials. Opt Quant Electron 2017;49(2):1–15.
[33] Zhang Z. Abundant exact traveling wave solutions for the Klein-Gordon-Zakharov equations via the tanh-coth expansion method and Jacobi elliptic function expansion method. Rom Journ Phys 2013;58(7–8):749–65.
[34] Zhang Z. Jacobi elliptic function expansion method for the mKdVZK and the Hirota equations. Rom Journ Phys 2015;60(9–10):1384–94.
[35] Zhang Z. First integral method and exact solutions to nonlinear partial differential equations arising in mathematical physics. Rom Rep Phys 2013;65(4):1155–69.
[36] He JH. Variational principle and periodic solution of the kundu–mukherjee–naskar equation. Results Phys 2020;17:103031.
[37] Ullah N, Asjad MI, Awrejcewicz J, Muhammad T, Baleanu D. On soliton solutions of fractional-order nonlinear model appears in physical sciences. AIMS Mathematics 2022;7(5):7421–40.
[38] Razzaq W, Zafar A, Ahmed HM, Rabied WB. Construction solitons for fractional nonlinear Schrödinger equation with β-time derivative by the new sub-equation method. J Ocean Engi Sci 2022. https://doi.org/10.1016/j.joes.2022.06.013.
[39] Siddique I, Jaradat MMM, Zafar A, Bukht Mehdi K, Osman MS. Exact traveling wave solutions for two prolific conformable M-Fractional differential equations via three diverse approaches. Results Phys 2021;28:104557.
[40] Zhang X-Z, Siddique I, Bukht Mehdi K, Elmandouh AA, Inc M. Novel exact solutions, bifurcation of nonlinear and supernonlinear traveling waves for M-fractional generalized reaction Duffing model and the density dependent M-fractional diffusion reaction equation. Results Phys 2022;37:105485.
[41] Seadawy AR, Ali A, Albarakati WA, Baleanu D. Propagation of traveling wave solutions to the Vekhnenko-Parkes dynamical equation via modified mathematical methods. Appl Math J Chinese Univ 2022;37(1):21–34.
[42] Arshed S. New soliton solutions to the perturbed Nonlinear Schrodinger equation by exp expansion method. Optik 2020;220:165123.
[43] Hosseini K, Ayati Z, Ansari R. New exact solutions of the Tzitzeica-type-equations in non-linear optics using the function method. Journel of Modern Optics 2018;65(7):847–51.
[44] Alabedalhadi M, Al-Smadi M, Al-Omari S, Baleanu D, Momani S. Structure of optical soliton solution for nonlinear resonant Space-Time Schrodinger equation in conformable sense with full nonlinearity term. Phys Scr 2020;95(10):105215.
[45] Baskonus HM, Bulut H, Sulaıman TA. Investigation of various travelling wave solutions to the extended (2+1)- dimensional quantum ZK equation. European Physical Journal Plus 2017;132:482.
[46] Bulut H, Sulaıman TA, Baskonus HM. On the exact solutions to some system of complex nonlinear models. Applied Mathematics and Nonlinear Sciences 2021;6(1):29–42.

[47] Z. Zhang X.Y. Gan D.M. Yu Bifurcation behavior of the traveling wave solutions of nonlinear the perturbed nonlinear Schrodingers equation with Kerr law nonlinearity Zeitschrift fur Naturforschung A 2011 66a(12):721–727.
[48] Zhou Q, Sonmezoglu A, Ekici M, Mirzazadeh M. Optical solitons of some fractional differential equations in nonlinear optics. J Mod Opt 2017;64:2345–9.
[49] Pinar Z, Öziş T. Solutions of modified equal width equation by means of the auxiliary equation with a sixth degree nonlinear term. Lecture Notes in Electrical Engineering 2013;185:139–48.
[50] Su D, Tang SQ. The sine-cosine method for generalized KdV equation with generalized evolution. Far East Journal of Applied Mathematics 2012;62:117–28.
[51] Yang S, Xu T. Steady bifurcation and solitary waves of modified equal width equation. European Physical Journal Plus 2017;132:369.
[52] Lu D, Seadawy AR, Ali A. Dispersive traveling wave solutions of the Equal-Width and modified equal-Width equations via mathematical methods and its applications. Results Phys 2018;9:313–20.
[53] Raslan K, Ali KK, Shallal MA. The modified extended tanh method with the Riccati equation for solving the space-time fractional EW and MEW equations. Chaos, Solitons Fractals 2017;103:404–9.
[54] Korkmaz A. Exact solutions of space-time fractional EW and modified EW equations. Chaos, Solitons Fractals 2017;96:132–8.
[55] Lu D, Ye S. Optical solitary wave solutions of the space-time fractional modified equal-width equation and their applications. Int J Math Res 2019;8(1):1–20.
[56] Ali U, Mastoi S, Othman WAM, Khater MMA, Sohail M. Computation of traveling wave solution for nonlinear variable-order fractional model of modified equal width equation. AIMS Mathematics 2021;6(9):10055–69.
[57] Demiray S, Unsal O, Bekir A. New exact solutions for Boussinesq type equations by using and expansion method. Acta Phys Pol A 2014;125(5):1093–8.
[58] Zhang Y, Zhang L, Pang J. Application of expansion method for solving Schrodinger's equation with three-order dispersion. Adv Appl Math 2017;6(2): 212–7.
[59] Shi D, Zhang Y. Diversity of exact solutions to the conformable space-time fractional MEW equation. Appl Math Lett 2019;99:105994.
[60] Elmandouh AA, Ibrahim AG. Bifurcation and travelling wave solutions for a (2+1)-dimensional KdV equation. J Taibah Uni Sci 2020;14(1):139–47.
[61] Elmandouh AA. Bifurcation and new traveling wave solutions for the 2D Ginzbur-Landau equation. European Physical Journal Plus 2020;135:648.
[62] Nuwairan MA, Elmandouh AA. Qualitative analysis and wave propagation of the nonlinear model for low-pass electrical transmission lines. Phys Scr 2021;96(9): 095214.
[63] Elbrolosy ME, Elmandouh AA. Dynamical behaviour of nondissipative double dispersive microstrain wave in the microstructured solids. European Phys J Plus 2021;136(9):1–20.
[64] Elbrolosy ME. Qualitative analysis and new soliton solutions for the coupled nonlinear Schrödinger type equations. Phys Scr 2021;96(12):125275.
[65] Nemytskii V, Stepanov V. Qualitative Theory of Differential Equations. New York: Dover; 1989.