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On minimum cycle bases of the wreath product of wheels with stars

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Abstract**—**The length of a cycle basis of a graph is the sum of the lengths of its elements. A minimum cycle basis is a cycle basis with minimum length. In this work, a construction of a minimum cycle basis for the wreath product of wheels with stars is presented. Moreover, the length of minimum cycle basis and the length of its longest cycle are calculated.

Keywords**—**cycle space, minimum cycle basis, wreath product.

I. INTRODUCTION

For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. The set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y\backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \varnothing$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph G the dimension of the cycle space is the *cyclomatic number* or the *first Betti number*, dim $\mathcal{C}(G) = |E(G)| - |V(G)| + 1$ (see [4]).

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. The *length*, $|C|$, of the element C of the cycle space $\mathcal{C}(G)$ is the number of its edges. The *length*, $l(\mathcal{B})$, of a cycles basis $\mathcal{B}$ is the sum of the lengths of its elements: $l(\mathcal{B}) = \sum_{C\in\mathcal{B}} |C|$. A *minimum cycle basis* (MCB) is a cycle basis with minimum length. The length of the longest element in an MCB is denoted by $\lambda(G)$. Since the cycle space is a matroid in which an element C has weight $|C|$, the greedy algorithm can be used to extract an MCB (see [9]). The following results will be used frequently in the sequel (see [6]).

Lemma 2.2. (Jaradat, et al.) *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and that $E(A) \cap E(B)$ induces a forest in G (we allow the possibility that $E(A) \cap E(B) = \varnothing$). Then $A \cup B$ is linearly independent.*

In the present article we continue what we initiate in [1], [4], and [7] by studying the problem of constructing an MCB for the wreath product of wheels and stars, where an MCB of graphs find applications in sciences and engineering: for examples, biochemistry, structural engineering, surface reconstruction and public transportations (See [2], [3] and [8]).

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The Cartesian product $G\square H$ has the vertex set $V(G\square H) = V(G) \times V(H)$ and the edge set $E(G\square H) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$. (2) The Lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)\}$. (3) The wreath product $G\rho H$ has the vertex set $V(G\rho H) = V(G) \times V(H)$ and the edge set $E(G\rho H) = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2$ and $v_1v_2 \in H$, or $u_1u_2 \in G$ and there is $\alpha \in$Aut(H) such that $\alpha(v_1) = v_2\}$.

In the rest of this paper, we let $\{u_1, u_2, \dots, u_n\}$ be the vertex set of the wheel W_n(the star S_n), with $d_{W_n}(u_1) = n-1$ and $\{v_1, v_2, \dots, v_m\}$ be the vertex set S_m with $d_{S_m}(v_1) = m-1$. Moreover, N_{m-1} stands for the null graph with vertex set $\{v_2, v_3, \dots, v_m\}$. Wherever they appear a, b, c, d and l stand for vertices and $E(B) = \cup_{C\in B} E(C)$ where $B \subseteq \mathcal{C}(G)$.

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II. THE MINIMUM CYCLE BASIS OF $W_n\rho S_m$.

In this section, we present a minimum cycle basis of $W_n\rho S_m$. To proceed in our work we set the following sets of cycles:

$$\mathcal{R}_{lab} = \Big\{\mathcal{R}_{lab}^{(j)} = (l, v_{j+1})(a, v_j)(b, v_j)(l, v_{j+1}) | \; 2 \leq j \leq m-1\Big\},$$

$$\mathcal{M}_{lab} = \Big\{\mathcal{M}_{lab}^{(j)} = (l, v_{j+1})(a, v_j)(b, v_{j+1})(l, v_{j+1}) \mid \; 2 \leq j \leq m-1\Big\}.$$

Also, for $j = 1, 2, \dots, m$ we set the following cycle

$$\mathcal{U}_{lab}^{(j)} = (l, v_j)(a, v_j)(b, v_j)(l, v_j).$$

Let

$$\mathcal{U}_{lab} = \cup_{j=2}^{m} \mathcal{U}_{lab}^{(j)}$$

Lemma 2.1. *$\mathcal{S}_{lab} = \mathcal{U}_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{R}_{bal} \cup \mathcal{R}_{abl} \cup \mathcal{M}_{lab} \cup \mathcal{M}_{lba}$ is a linearly independent set.*

Proof. Since each of $\mathcal{U}_{lab}, \mathcal{R}_{lab}, \mathcal{M}_{lab}, \mathcal{M}_{lba}, \mathcal{R}_{bal}$, and $\mathcal{R}_{abl}$ consists of edge disjoint cycles, as a result each of which is linearly independent. Now, any linear combination of cycles of $\mathcal{R}_{lab}$ contains an edge of the form $(l, v_{i+1})(a, v_i)$ for some $2 \leq i \leq m-1$ which occurs in no cycle of $\mathcal{U}_{lab}$. Thus $\mathcal{U}_{lab} \cup \mathcal{R}_{lab}$ is linearly independent. Any linear combination of cycles of $\mathcal{M}_{lab}$ contains an edge of the form $(b, v_{i+1})(a, v_i)$ for some $2 \leq i \leq m-1$ which does not occur in any cycle of $\mathcal{U}_{lab} \cup \mathcal{R}_{lab}$. Hence $\mathcal{U}_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{M}_{lab}$ is linearly independent. Similarly, since any linear combination of cycles of $\mathcal{M}_{lba}$ contains an edge of the form $(a, v_{i+1})(b, v_i)$ for some

$2 \le i \le m-1$ which occurs in no cycle of $\mathcal{U}_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{M}_{lab}$, as a result $\mathcal{U}_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{M}_{lab} \cup \mathcal{M}_{lba}$ is linearly independent. Also, since any linear combination of cycles of $\mathcal{R}_{bal}$ contains an edge of the form $(b, v_{i+1})(l, v_i)$ for some $2 \le i \le m-1$ which does not belong to any cycle of $\mathcal{U}_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{M}_{lab} \cup \mathcal{M}_{lba}$, we have $\mathcal{U}_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{M}_{lab} \cup \mathcal{M}_{lba} \cup \mathcal{R}_{bal}$ is linearly independent. Finally, any linear combination of cycles of $\mathcal{R}_{abl}$ contains an edge of the form $(a, v_{i+1})(l, v_i)$ for some $2 \le i \le m-1$ which does not appear in any cycle of $\mathcal{U}_{lab} \cup \mathcal{R}_{lab} \cup \mathcal{M}_{lab} \cup \mathcal{M}_{lba} \cup \mathcal{R}_{bal}$. Therefore, $\mathcal{S}_{lab}$ is linearly independent. □

Now, for each $2 \le j \le m-2$, we define the following set of cycles:

$$\mathcal{E}^{(j)}_{lab} = \left\{ \mathcal{E}^{(j,i)}_{lab} = (l, v_{i+j-1})(a, v_i)(b, v_{i+j})(l, v_{i+j-1}) \mid 2 \le i \le m-j \right\}.$$

Let

$$\mathcal{E}_{lab} = \cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{lab}$$

Lemma 2.2. *The set* $\mathcal{Z}^{(j)}_{lab} = \mathcal{E}^{(j)}_{lba} \cup \mathcal{E}^{(j)}_{lab} \cup \mathcal{E}^{(j)}_{abl} \cup \mathcal{E}^{(j)}_{alb} \cup \mathcal{E}^{(j)}_{bla} \cup \mathcal{E}^{(j)}_{bal}$ *is linearly independent.*

Proof. Since each of $\mathcal{E}^{(j)}_{lba}$, $\mathcal{E}^{(j)}_{lab}$, $\mathcal{E}^{(j)}_{abl}$, $\mathcal{E}^{(j)}_{alb}$, $\mathcal{E}^{(j)}_{bla}$ and $\mathcal{E}^{(j)}_{bal}$ consists only of edge disjoint cycles and since

$$\begin{aligned} E(\mathcal{E}^{(j)}_{lab}) \cap E(\mathcal{E}^{(j)}_{lba}) &= E(\mathcal{E}^{(j)}_{lab}) \cap E(\mathcal{E}^{(j)}_{abl}) \\ &= E(\mathcal{E}^{(j)}_{lba}) \cap E(\mathcal{E}^{(j)}_{abl}) \\ &= \varnothing, \end{aligned}$$

$\mathcal{E}^{(j)}_{lba} \cup \mathcal{E}^{(j)}_{lab} \cup \mathcal{E}^{(j)}_{abl}$ is linearly independent by Lemma 1.2. Any linear combination of cycles of $\mathcal{E}^{(j)}_{alb}$ contains an edge of the form $(l, v_i)(b, v_{i+j})$ for some $2 \le i \le m-j$ which does not occur in any cycle of $\mathcal{E}^{(j)}_{lba} \cup \mathcal{E}^{(j)}_{lab} \cup \mathcal{E}^{(j)}_{abl}$. Hence, $\mathcal{E}^{(j)}_{lba} \cup \mathcal{E}^{(j)}_{lab} \cup \mathcal{E}^{(j)}_{abl} \cup \mathcal{E}^{(j)}_{alb}$ is linearly independent. Since any linear combination of cycles of $\mathcal{E}^{(j)}_{bla}$ contains an edge of the form $(l, v_i)(a, v_{i+j})$ for some $2 \le i \le m-j$ which does not appear in any cycle of $\mathcal{E}^{(j)}_{lba} \cup \mathcal{E}^{(j)}_{lab} \cup \mathcal{E}^{(j)}_{abl} \cup \mathcal{E}^{(j)}_{alb}$. Thus $\mathcal{E}^{(j)}_{lba} \cup \mathcal{E}^{(j)}_{lab} \cup \mathcal{E}^{(j)}_{abl} \cup \mathcal{E}^{(j)}_{alb} \cup \mathcal{E}^{(j)}_{bla}$ is linearly independent. Similarly, any linear combination of cycles of $\mathcal{E}^{(j)}_{bal}$ contains an edge of the form $(a, v_i)(l, v_{i+j})$ for some $2 \le i \le m-j$ which belongs to no cycle of $\mathcal{E}^{(j)}_{lba} \cup \mathcal{E}^{(j)}_{lab} \cup \mathcal{E}^{(j)}_{abl} \cup \mathcal{E}^{(j)}_{alb} \cup \mathcal{E}^{(j)}_{bla}$. Therefore, $\mathcal{Z}^{(j)}_{lab}$ is linearly independent. □

Lemma 2.3. *Any linear combination of cycles of* $\cup_{j=2}^{m-2} \mathcal{Z}^{(j)}_{lab}$ *contains at least one edge of the following forms:* $(a, v_i)(b, v_{i+j})$, $(b, v_i)(a, v_{i+j})$, $(b, v_i)(l, v_{i+j})$, $(l, v_i)(b, v_{i+j})$, $(l, v_i)(a, v_{i+j})$ *and* (a, v_i) (l, v_{i+j}) *where* $2 \le j \le m-2$ *and* $2 \le i \le m-j$.

Proof. Assume that C is a linear combination of the cycles of $\mathcal{O} = \{O_1, O_2, \ldots, O_r\} \subseteq \cup_{j=2}^{m-2} \mathcal{Z}^{(j)}_{lab}$ and C does not contain any edge of the above stated forms. Then we split our work into the following cases:

Case 1: $\mathcal{O}$ contains at least one cycle of $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}$, say $\mathcal{E}^{(j_0,i_0)}_{alb} = Z_1$. Note that $e = (l, v_{i_0})(b, v_{i_0+j_0}) \in E(\mathcal{E}^{(j_0,i_0)}_{alb})$ which does not occur in any other cycle of $\cup_{j=2}^{m-2} \mathcal{Z}^{(j)}_{lab}$. Thus, e does not belong to any other cycle of $\mathcal{O}$. Therefore $e \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$. This contradicts the fact that $e \notin E(C)$.

Case 2: $\mathcal{O}$ contains no cycle of $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}$, but contains at least one cycle of $\cup_{j=2}^{m-1} \mathcal{E}^{(j)}_{bla}$, say $\mathcal{E}^{(j_0,i_0)}_{bla} = O_1$. As in case one, $e = (l, v_{i_0})(a, v_{i_0+j_0}) \in E(\mathcal{E}^{(j_0,i_0)}_{bla})$ which does not occur in any other cycle of $(\cup_{j=2}^{m-2} \mathcal{Z}^{*(j)}_{lab}) - (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb})$. Thus, e does not belong to any other other cycle of $\mathcal{O}$. Therefore, $e \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$. This contradicts the fact that $e \notin E(C)$.

Case 3: $\mathcal{O}$ contains no cycle of $(\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla})$, but contains at least one cycle of $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{lab}$ or $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bal}$. Then we consider the following subcases:

Subcase 3.a: $\mathcal{O}$ contains cycle of $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{lab}$. Let j_0 $(2 \le j_0 \le m-2)$ be the largest integer such that $\mathcal{E}^{(j_o)}_{lab} \cap \mathcal{O} \ne \varnothing$. Let i_0 $(2 \le i_0 \le m-j_0)$ be the largest integer greater such that $\mathcal{E}^{(j_0,i_0)}_{lab} \in \mathcal{O}$, say $Z_1 = \mathcal{E}^{(j_0,i_0)}_{lab}$. Note that $e_1 = (a, v_{i_0})(b, v_{i_0+j_0}) \in E(O_1)$ and $e_1 \notin E(C)$. Moreover, the only other cycle of $(\cup_{j=2}^{m-2} \mathcal{Z}^{(j)}_{lab}) - (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla})$ that contains e_1 is $\mathcal{E}^{(j_0+1,i_0)}_{bal}$. Hence, $\mathcal{E}^{(j_0+1,i_0)}_{bal} \in \mathcal{O}$, say $O_2 = \mathcal{E}^{(j_0+1,i_0)}_{bal}$. Note that $e_2 = (a, v_{i_0})(l, v_{i_0+(j_0+1)}) \in E(Z_2)$ which does not occur in O_1. Thus, $e_2 \in E(O_1 \oplus O_2)$. Since $e_2 \notin E(C)$ and $\mathcal{E}^{(j_0+2,i_0)}_{lab}$ is the only other cycle of $(\cup_{j=2}^{m-2} \mathcal{Z}^{(j)}_{lab}) - (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla})$ which contains e_2, $\mathcal{E}^{(j_0+2,i_0)}_{lab} \in \mathcal{O}$, say $O_3 = \mathcal{E}^{(j_0+2,i_0)}_{lab}$. Note that $e_3 = (a, v_{i_0})(b, v_{i_0+(j_0+2)}) \in E(O_3)$ which does not occur in $O_1 \oplus O_2$. Thus, $e_3 \in E(O_1 \oplus O_2 \oplus O_3)$. Now, $e_3 \notin E(C)$ and $\mathcal{E}^{(j_0+3,i_0)}_{lab}$ is the only other cycle of $(\cup_{j=2}^{m-2} \mathcal{Z}^{(j)}_{lab}) - (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla})$ which contains e_2. By continuing in this process, we get the following: There is r_0 such that $O_{r_0} = \mathcal{E}^{(m-i_0,i_0)}_{lab} \in \mathcal{O}$ or $O_{r_0} = \mathcal{E}^{(m-i_0,i_0)}_{bal} \in \mathcal{O}$. Note that in both cases $e_{r_0} = (a, v_{i_0})(b, v_m) \in O_{r_0}$ which occurs in no other cycles of $(\cup_{j=2}^{m-2} \mathcal{Z}^{(j)}_{lab}) - (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla})$. Thus, $e_{r_0} \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$. This contradicts the fact that $e_{r_0} = (a, v_{i_0})(b, v_{i_0+(m-i_0)}) \notin E(C)$.

Subcase 3.b: $\mathcal{O}$ contains cycle of $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bal}$. The we use the same arguments as in Case 3.a to have a similar contradiction.

Case 4: $\mathcal{O}$ contains no cycle of $(\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{lab}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bal})$ but contains at least one cycle of $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{abl}$. As in Case 3.a, let j_0 $(2 \le j_0 \le m-2)$ be the largest integer such that $\mathcal{E}^{(j_o)}_{abl} \cap \mathcal{O} \ne \varnothing$. Let i_0 $(2 \le i_0 \le m-j_0)$ be the largest integer greater such that $\mathcal{E}^{(j_0,i_0)}_{abl} \in \mathcal{O}$, say $O_1 = \mathcal{E}^{(j_0,i_0)}_{abl}$. Note that $e_1 = (b, v_{i_0})(l, v_{i_0+j_0}) \in E(O_1)$ and $e_1 \notin E(C)$. Also note that $\mathcal{E}^{(j_0+1,i_0)}_{lba}$ is the only other cycle of $(\cup_{j=2}^{m-2} \mathcal{Z}^{*(j)}_{lab}) - (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{lab}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bal})$ which contains e_1. Thus, $O_2 = \mathcal{E}^{(j_0+1,i_0)}_{lba} \in \mathcal{O}$. By continuing in the process as in Case 3, taking into account only Subcase 3.a, we get a similar contradiction.

Case 5: $\mathcal{O}$ contains no cycle of $(\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{alb}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bla}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{lab}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{bal}) \cup (\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{abl})$. Then $\mathcal{O}$ contains at least one cycle of $\cup_{j=2}^{m-2} \mathcal{E}^{(j)}_{lba}$, say $O_1 = \mathcal{E}^{(j_0,i_0)}_{lba}$. Then, $e = (b, v_{i_0})(l, v_{i_0+j_0}) \in E(O_1)$ and does not occur in any other cycle of $(\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)}) - (\cup_{j=2}^{m-2}\mathcal{E}_{alb}^{(j)}) \cup (\cup_{j=2}^{m-2}\mathcal{E}_{bla}^{(j)}) \cup (\cup_{j=2}^{m-2}\mathcal{E}_{lab}^{(j)}) \cup (\cup_{j=2}^{m-2}\mathcal{E}_{bal}^{(j)}) \cup (\cup_{j=2}^{m-2}\mathcal{E}_{abl}^{(j)})$. Thus, $e \in E(\bigoplus_{i=1}^{r} O_i)$. This contradicts the fact that $e \notin E(C)$. □

Remark 2.4. *By using same arguments as in Lemma 4.3, we can show, by taking into account Cases 1, 4, and 5 and Subcase 3.a, that any linear combination of* $(\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)}) - (\cup_{j=2}^{m-2}(\mathcal{E}_{bla}^{(j)} \cup \mathcal{E}_{bal}^{(j)}))$ *contains at least one edge of the following forms:* $(a, v_i)(b, v_{i+j}), (b, v_i)(a, v_{i+j}), \ (b, v_i)(l, \ v_{i+j})$, *and* $(l, v_i)(b, v_{i+j})$ *where* $2 \le j \le m-2$ *and* $2 \le i \le m-j$. □

Lemma 2.5. *The set* $\mathcal{S}_{lab} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)})$ *is linearly independent.*

Proof. We use mathematical induction on m to show that $(\cup_{j=2}^{m-2}\mathcal{Z}_{ab}^{(j)})$ is linearly independent. If $m = 4$, then $(\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)})$ consists only of $\mathcal{Z}_{lab}^{(2)}$. Thus, the result follows from Lemma 2.2. Assume that m is grater than 4 and the result is true for less than m. Note that $(\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)}) = (\cup_{j=2}^{m-3}\mathcal{Z}_{lab}^{(j)}) \cup \mathcal{Z}_{lab}^{(m-2)}$. By a similar argument to that in proof of Lemma 2.3, any linear combination of cycles of $\mathcal{Z}_{lab}^{(m-2)}$ contains at least one edge of the following: $(a, v_2)(b, v_m), (b, v_2)(a, v_m), (b, v_2)(l, v_m), (l, v_2)(b, v_m)$, $(l, v_2)(a, v_m)$, and $(a, v_2)(l, v_m)$. Since non of them occurs in any cycle of $(\cup_{j=2}^{m-3}\mathcal{Z}_{lab}^{(j)})$, $(\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)})$ is linearly independent. By Lemma 2.1 $\mathcal{S}_{lab}$ is linearly independent. Now, by Lemma 2.3 any linear combination of cycles of $(\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)})$ contains at least one edge of the following forms: $(a, v_i)(b, v_{i+j}), (b, v_i)(a, v_{i+j}), (b, v_i)(l, v_{i+j}), (l, v_i)(b, v_{i+j}), (l, v_i)(a, v_{i+j})$, and $(a, v_1)(l, v_{i+j})$ for some $2 \le i \le m-j$ and $2 \le j \le m-2$, which does not occur in any cycle of $\mathcal{S}_{lab}$. Therefore, $\mathcal{S}_{lab} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)})$ is linearly independent. □

Let

$$\mathcal{S}_{lab}^{*} = (\mathcal{S}_{lab} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{lab}^{(j)})) - (\mathcal{R}_{abl} \cup (\cup_{j=2}^{m-2}(\mathcal{E}_{bla}^{(j)} \cup \mathcal{E}_{bal}^{(j)}))).$$

Lemma 2.6. *Any linear combination of cycles of* $\mathcal{S}_{lab} - \mathcal{R}_{abl}$ *contains at least one edge of the following forms:* $(l, v_{i+1})(b, v_i), (l, v_i)(b, v_{i+1}), (l, v_i)(b, v_i), (a, v_{i+1})(b, v_i)$ *and* $(a, v_i)(b, v_{i+1})$.

Proof. Let C be a linear combination of the cycles of $\mathcal{O} = \{O_1, O_2, \ldots, O_r\} \subseteq \mathcal{S}_{lab} - \mathcal{R}_{abl}$. Then we consider the following cases:

Case 1: $\mathcal{O}$ contains at least one cycle of $\mathcal{M}_{lba}$, say $O_1 = \mathcal{M}_{lba}^{(i)}$. Note that $e = (a, v_{i+1})(b, v_i) \in O_1$ and does not occur in other cycles of $\mathcal{S}_{lab}^{*}$. And so e does not occur in any other cycle of $\mathcal{S}_{lab}$ Thus, $e \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$.

Case 2: $\mathcal{O}$ contains no cycle of $\mathcal{M}_{lba}$, but contains at least one cycle of $\mathcal{R}_{lab}$, say $O_1 = \mathcal{R}_{lab}^{(i)}$. Note that $e = (l, v_{i+1})(b, v_i) \in E(O_1)$ and does not occur in other cycles of $\mathcal{S}_{lab}^{*} - \mathcal{M}_{lba}$. And so e does not occur in any other cycle of $\mathcal{S}_{lab} - \mathcal{M}_{lba}$. Thus, $(l, v_{i+1})(b, v_i) \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$.

Case 3: $\mathcal{O}$ contains no cycle of $\mathcal{M}_{lba} \cup \mathcal{R}_{lab}$, but contains at least one cycle of $\mathcal{R}_{bal}$, say $O_1 = \mathcal{R}_{bal}$. Note that $e = (l, v_i)(b, v_{i+1}) \in E(O_1)$ and does note occur in other cycles of $\mathcal{S}_{lab}^{*} - \mathcal{M}_{lba} \cup \mathcal{R}_{lab}$. And so e does not occur in any other cycle of $\mathcal{S}_{lab} - \mathcal{M}_{lba} \cup \mathcal{R}_{lab}$. Thus, $(l, v_i)(b, v_{i+1}) \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$.

Case 4: $\mathcal{O}$ contains no cycle of $\mathcal{M}_{lba} \cup \mathcal{R}_{lab} \cup \mathcal{R}_{bal}$, but contains at least one cycle of $\mathcal{M}_{lab}$, say $O_1 = \mathcal{M}_{lab}^{(i)}$. Note that $e = (a, v_i)(b, v_{i+1}) \in O_1$ and does not occur in other cycles of $\mathcal{S}_{lab}^{*} - \mathcal{M}_{lba} \cup \mathcal{R}_{lab} \cup \mathcal{R}_{bal}$. And so e does not occur in any other cycle of $\mathcal{S}_{lab} - \mathcal{M}_{lba} \cup \mathcal{R}_{lab} \cup \mathcal{R}_{bal}$. Thus, $(a, v_i)(b, v_{i+1}) \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$.

Case 5: $\mathcal{O}$ contains no cycle of $\mathcal{M}_{lba} \cup \mathcal{R}_{lab} \cup \mathcal{R}_{bal} \cup \mathcal{M}_{laa}$. Then $\mathcal{O}$ must contains at least one cycle of $\mathcal{U}_{lab}$, say $O_1 = \mathcal{U}_{lab}^{(i)}$. Note that $e = (l, v_i)(b, v_i) \in O_1$ and does not occur in other cycle of $\mathcal{S}_{lab}^{*} - \mathcal{M}_{lba} \cup \mathcal{R}_{lab} \cup \mathcal{R}_{bal} \cup \mathcal{M}_{lba}$. And so e does not occur in any other cycle of $\mathcal{S}_{lab} - \mathcal{M}_{lba} \cup \mathcal{R}_{lab} \cup \mathcal{R}_{bal} \cup \mathcal{M}_{lba}$. Thus, $(l, v_i)(b, v_i) \in E(\bigoplus_{i=1}^{r} O_i) = E(C)$. □

Lemma 2.7. $\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*}$ *is linearly independent set.*

Proof. We prove that $\cup_{i=3}^{n-1}\mathcal{S}_{u_1 u_i u_{i+1}}^{*}$ is linearly independent using mathematical induction on n. If $n = 4$, then $\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*} = \mathcal{S}_{u_1 u_3 u_4}^{*}$ which is linearly independent by Lemma 2.5. Assume that n is grater than 4 and the result is true for less than n. Note that $\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*} = (\cup_{i=3}^{n-2} \mathcal{S}_{u_1 u_i u_{i+1}}^{*}) \cup \mathcal{S}_{u_1 u_{n-1} u_n}^{*}$. By combining Lemmas 2.3 and 2.6, any linear combination of cycles of $\mathcal{S}_{u_1 u_{n-1} u_n}^{*}$ contains an edge of the form $(u_{n-1}, v_i)(u_n, v_k)$ or $(u_1, v_i)(u_n, v_k)$ for $2 \le i, k \le m$ which does not occur in any cycle of $(\cup_{i=3}^{n-2} \mathcal{S}_{u_1 u_i u_{i+1}}^{*})$. Thus, $\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*}$ is linearly independent. □

Let $\mathcal{F} = \cup_{i=1}^{n} V(W_n) \Box S_m$ and $\mathcal{L} = W_n \Box v_1$. Then, $W_n \rho S_m = \mathcal{F} \cup \mathcal{L} \cup \mathcal{K}$ where $\mathcal{K} = W_n \rho S_m - E(\mathcal{F} \cup \mathcal{L})$. Note that $|E(\mathcal{K})| = |E(W_n[N_{m-1}]| = 2(m-1)^2(n-1)$. Thus,

$$\dim \mathcal{C}(\mathcal{K}) = 2(m-1)^2(n-1) - n(m-1) + 1 \qquad (1)$$

Lemma 2.8. *The set* $\mathcal{B}(\mathcal{K}) = (\mathcal{S}_{u_1 u_2 u_3} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{u_1 u_2 u_3}^{(j)})) \cup (\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*}) \cup (\mathcal{U}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_2 u_n} \cup \mathcal{E}_{u_1 u_n u_2} \cup \mathcal{E}_{u_1 u_2 u_n})$ *is a cycle basis of* $\mathcal{K}$.

Proof. We know that $\mathcal{S}_{u_1 u_2 u_3} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{u_1 u_2 u_3}^{(j)})$ and $(\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*})$ are linearly independent sets by Lemmas 2.5 and 2.7. Note that $\mathcal{U}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_2 u_n} \cup \mathcal{E}_{u_1 u_n u_2} \cup \mathcal{E}_{u_1 u_2 u_n} \subseteq \mathcal{S}_{u_1 u_n u_2} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{u_1 u_n u_2}^{(j)})$. Thus, $\mathcal{U}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_2 u_n} \cup \mathcal{E}_{u_1 u_n u_2} \cup \mathcal{E}_{u_1 u_2 u_n}$ is linearly independent by Lemma 2.5. By same arguments as in the proof of Lemmas 2.3 and 2.6, we get that any linear combination of cycles of $\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*}$ contains an edge of the form $(u_i, v_j)(u_{i+1}, v_k)$ or $(u_1, v_j)(u_{i+1}, v_k)$ for $2 \le j, k \le m$ and $3 \le i \le n-1$. Note that non of the above forms occurs in any cycle of $\mathcal{S}_{u_1 u_2 u_3} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{u_1 u_2 u_3}^{(j)})$. Thus, $(\mathcal{S}_{u_1 u_2 u_3} \cup (\cup_{j=2}^{m-2}\mathcal{Z}_{u_1 u_2 u_3}^{(j)})) \cup (\cup_{i=3}^{n-1} \mathcal{S}_{u_1 u_i u_{i+1}}^{*})$ is linearly independent. Similarly, by using arguments as in Lemmas 2.3 and 2.6, we have that any linear combination of cycles of $\mathcal{U}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_n u_2} \cup \mathcal{M}_{u_1 u_2 u_n} \cup \mathcal{E}_{u_1 u_n u_2} \cup \mathcal{E}_{u_1 u_2 u_n}$ contains an edge of the following forms: $(u_n, v_i)(u_2, v_i), (u_n, v_{j+1})$

$(u_2, v_j), (u_2, v_{j+1})(u_n, v_j), (u_2, v_k)(u_n, v_{k+j})$ and $(u_n, v_k)(u_2, v_{k+j})$ for $2 \leq i \leq m, 2 \leq j \leq m-1$ and $2 \leq k \leq m-j$ which does not occur in any cycle of $(\mathcal{S}_{u_1u_2u_3} \cup (\cup_{j=2}^{m-2} \mathcal{Z}_{u_1u_2u_3}^{(j)})) \cup (\cup_{i=3}^{n-1} \mathcal{S}^*_{u_1u_iu_{i+1}})$. Thus, $\mathcal{B}(\mathcal{K})$ is linearly independent. Now,

$$\begin{aligned} |\mathcal{S}_{u_1u_2u_3}| &= |\mathcal{S}_{lab}| = |\mathcal{U}_{lab}| + 3|\mathcal{R}_{lab}| + 2|\mathcal{M}_{lba}| \\ &= (m-1) + 3(m-2) + 2(m-2) \\ &= 6m - 11, \end{aligned} \tag{2}$$

and

$$\begin{aligned} |\mathcal{E}_{u_1u_nu_2}| &= |\mathcal{E}_{lab}| \\ &= \sum_{j=2}^{m-2} |\mathcal{E}_{lab}^{(j)}| \\ &= \sum_{j=2}^{m-2} (m-j-1) \\ &= \frac{(m-3)(m-2)}{2} \\ &= \frac{m^2 - 5m + 6}{2}. \end{aligned} \tag{3}$$

Thus,

$$\begin{aligned} |\cup_{j=2}^{m-2} \mathcal{Z}_{u_1u_2u_3}^{(j)}| &= 6\ |\mathcal{E}_{lab}| \\ &= 6(\frac{m^2-5m+6}{2}) \\ &= 3m^2 - 15m + 18, \end{aligned} \tag{4}$$

and

$$\begin{aligned} |\mathcal{S}^*_{u_1u_iu_{i+1}}| &= |\mathcal{S}^*_{lab}| \\ &= |\mathcal{S}_{lab}| + \ |\cup_{j=2}^{m-2} \mathcal{Z}_{lab}^{*(i)}| - (|\mathcal{R}_{lab}| + 2\ |\mathcal{E}_{lab}|) \\ &= (3m^2 - 9m + 7) - \\ &\quad ((m-2) + m^2 - 5m + 6) \qquad (5) \\ &= 2m^2 - 5m + 3. \qquad (6) \end{aligned}$$

But

$$\begin{aligned} |\mathcal{B}(\mathcal{K})| &= |\mathcal{S}_{u_1u_2u_3}| + |\cup_{j=2}^{m-2} \mathcal{Z}_{u_1u_2u_3}^{(j)}| \\ &\quad + |\cup_{j=3}^{m-2} \mathcal{S}^*_{u_1u_iu_{i+1}}| + |\mathcal{U}_{u_1u_nu_2}| \\ &\quad + 2|\mathcal{M}_{u_1u_nu_2}| + 2|\mathcal{E}_{u_1u_nu_2}|. \end{aligned}$$

Hence by equations (2)-(5),

$$\begin{aligned} |\mathcal{B}(\mathcal{K})| &= (6m-11) + (3m^2 - 15m + 18) \\ &\quad + \sum_{i=3}^{n-1} (2m^2 - 5m + 3) + \\ &\quad (m-1) + 2(m-2) + 2(\frac{m^2-5m+6}{2}) \\ &= 2(m-1)^2(n-1) - n(m-1) + 1 \\ &= \dim(\mathcal{C}(\mathcal{K})). \end{aligned}$$

Therefore, $\mathcal{B}(\mathcal{K})$ is a cycle basis for $\mathcal{K}$. □

Now, for each $i = 2, 3, \ldots, n$ consider the following sets of cycles:

$$\mathcal{N}_{u_i} = \left\{ \mathcal{N}_{u_i}^{(j)} = (u_1, v_1)(u_i, v_1)(u_i, v_j)(u_1, v_2)(u_1, v_1) | \; 2 \leq j \leq m \right\}.$$

Also, set

$$\mathcal{T} = \left\{ \mathcal{T}^{(j)} = (u_1, v_1)(u_1, v_j)(u_2, v_2)(u_2, v_1)(u_1, v_1) \; | \; 3 \leq j \leq m \right\}.$$

Let

$$\mathcal{U}^{(1)} = \cup_{i=2}^{n} \mathcal{U}_{u_1u_iu_{i+1}}^{(1)} \text{ and } \mathcal{N} = \cup_{i=2}^{n} \mathcal{N}_{u_i}.$$

Lemma 2.9. *$\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}$ is a cycle basis of $\mathcal{K} \cup \mathcal{L}$ and so of $\mathcal{K} \cup \mathcal{L} \cup \{(u_1, v_1)(u_1, v_2)\}$.*

Proof. Note that $\mathcal{U}^{(1)}$ is the set of all triangle bounded faces of $W_n \Box u_1$. Thus, $\mathcal{U}^{(1)}$ is a cycles basis of $\mathcal{L}$. Since $\mathcal{B}(\mathcal{K})$ is a linearly independent set and $E(\mathcal{B}(\mathcal{K})) \cap E(\mathcal{U}^{(1)}) = \varnothing$, $\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}$ is linearly independent. Now,

$$\begin{aligned} |\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}| &= |\mathcal{B}(\mathcal{K})| + |\mathcal{U}^{(1)}| \\ &= 2(m-1)^2(n-1) - n(m-1) \\ &\quad + 1 + (n-1) \qquad (7) \\ &= 2(m-1)^2(n-1) - n(m-1) \\ &\quad + 2(n-1) - n + 2 \qquad (8) \\ &= \dim(\mathcal{K} \cup \mathcal{U}^{(1)}). \qquad (9) \end{aligned}$$

Thus, $\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}$ is a cycle basis of $\mathcal{K} \cup \mathcal{L}$. The second part follows from noting that the addition of $(u_1, v_1)(u_1, v_2)$ to the subgraph $\mathcal{K} \cup \mathcal{N}$ does not create any cycles. □

Remark 2.10. *Note that the addition of any edge of $\mathcal{F} - \{(u_1, v_1)(u_1, v_2)\}$ to the subgraph $\mathcal{K} \cup \mathcal{L} \cup \{(u_1, v_1)(u_1, v_2)\}$ creates at least one new cycle and the shortest cycle of $W_n \rho S_m$ contains the added edge is of length 4. Moreover, any cycle contains $(u_1, v_1)(u_1, v_2)$ must contain at least another edge of $\mathcal{F}$.* □

Lemma 2.11. *$\mathcal{N} \cup \mathcal{T}$ is linearly independent.*

Proof. We use mathematical induction on m to show that $\mathcal{N}_{u_i} = \cup_{j=2}^{m} \mathcal{N}_{u_i}^{(j)}$ is linearly independent. If $m = 2$, then $\mathcal{N}_{u_i}$ consists only of one cycle $\mathcal{N}_{u_i}^{(2)}$. Thus $\mathcal{N}_{u_i}$ is linearly independent. Assume that m is grater than 2 and the result is true for less than m. Note that $\mathcal{N}_{u_i} = (\cup_{j=2}^{m-1} \mathcal{N}_{u_i}^{(j)}) \cup \mathcal{N}_{u_i}^{(m)}$. Since the cycle $\mathcal{N}_{u_i}^{(m)}$ contains the edge $(u_i, v_1)(u_i, v_m)$ which occurs in no cycle of $\cup_{j=2}^{m-1} \mathcal{N}_{u_i}^{(j)}$, $\mathcal{N}_{u_i}$ is linearly independent for each i. Now, $E(\mathcal{N}_{u_i}) \cap E(\mathcal{N}_{u_j}) = \{(u_1, v_1)(u_1, v_2)\}$ whenever $i \neq j$. Therefore, $\mathcal{N}$ is linearly independent. Similarly we can show that $\mathcal{T}$ is linearly independent. Since $E(\mathcal{N}) \cap E(\mathcal{T}) = \{(u_1, v_1)(u_2, v_1), (u_2, v_1)(u_2, v_2)\}$ which is an edge set of a path, $\mathcal{N} \cup \mathcal{T}$ is linearly independent by Lemma 2.2. □

Theorem 4.12. *$\mathcal{B}(W_n \rho S_m) = \mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)} \cup \mathcal{N} \cup \mathcal{T}$ is a minimal cycle basis of $W_n \rho S_m$.*

Proof. By Lemmas 2.9 and 2.11 each of $\mathcal{N} \cup \mathcal{T}$ and $\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}$ is linearly independent. Note that any linear combination of cycles of $\mathcal{N} \cup \mathcal{T}$ must contains at least one edge of $\cup_{i=1}^{n} (u_i \Box S_m)$ which does not belong to any cycle of $\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}$. Thus, $\mathcal{B}(W_n \rho S_m)$ is linearly independent. Note that

$$|\mathcal{N}_{u_i}| = (m-1) \text{ and } |\mathcal{T}| = (m-2), \tag{10}$$

and,

$$|\mathcal{B}(W_n \rho S_m)| = |\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}| + |\mathcal{N}| + |\mathcal{T}|.$$

Thus by (2), (14) and (15), we get

$$\begin{aligned} |\mathcal{B}(W_n \rho S_m)| &= 2(m-1)^2(n-1) - n(m-1) + \\ & \quad 2(n-1) - n + 2 + \\ & \quad (n-1)(m-1) + (m-2) \\ &= (n-1)(2m^2 - 4m + 3) \\ &= \dim \mathcal{C}(W_n \rho S_m). \end{aligned}$$

Hence, $\mathcal{B}(W_n \rho S_m)$ is a cycle basis of $W_n \rho S_m$. Now, we show that $\mathcal{B}(W_n \rho S_m)$ is a minimum basis. Since any cycle of $\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}$ is of length 3 and $\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}$ is a basis for $W_n \rho S_m - E(\mathcal{F})$, as a result $|\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}|$ is the size of a maximum linearly independent set of $W_n \rho S_m - E(\mathcal{F})$ consisting of 3-cycles. But any cycle of $W_n \rho S_m$ that contains an edge of $\mathcal{F}$ must be of length at least 4. Therefore, $|\mathcal{B}(\mathcal{K}) \cup \mathcal{U}^{(1)}|$ is the size of a maximum linearly independent set of $W_n \rho S_m$ consisting of 3-cycles. Now, since each cycle of $\mathcal{N} \cup \mathcal{T}$ is of length 4 and since the cycle space is a matroid, as a result $\mathcal{B}(W_n \rho S_m)$ is a minimum cycle basis for $W_n \rho S_m$. □

Corollary 3.13 $l(W_n \rho S_m) = 8nm^2 - 17mn - 4m^2 + 8m + 5n - 13$, *and* $\lambda(W_n \rho S_m) = 4$. □

III. Conclusion

The works in this paper present some important results of the minimum cycle basis and cycle spaces of the wreath product of graphs. Some applications in sciences and engineering are indicated. In particular we have drawn attention to the use of MCB in biochemistry, structural engineering, surface reconstruction and public transportations (See [2], [3] and [8]).

References

[1] K.M. Al-Qeyyam and M.M.M. Jaradat, *On the basis number and the minimum cycle bases of the wreath product of some graphs II*, (To appear in JCMCC).

[2] L.O. Chua and L. Chen, *On optimally sparse cycles and coboundary basis for a linear graph*, IEEE Trans. Circuit Theory, 20, 54-76 (1973).

[3] G.M. Downs, V.J. Gillet, J.D. Holliday and M.F. Lynch, *Review of ring perception algorithms for chemical graphs*, J. Chem. Inf. Comput. Sci., 29, 172-187 (1989).

[4] F. Harary, "*Graph theory*", Addison-Wesley Publishing Co., Reading, Massachusetts, 1971.

[5] M.M.M. Jaradat, *On the basis number and the minimum cycle bases of the wreath product of some graphs I*, Discussiones Mathematicae Graph Theory 26, 113-134 (2006).

[6] M.M.M. Jaradat, M.Y. Alzoubi and E.A. Rawashdeh, *The basis number of the Lexicographic product of different ladders*, SUT Journal of Mathematics 40(2), 91-101 (2004).

[7] M.M.M. Jaradat and M.K. Al-Qeyyam, *On the basis number and the minimum cycle bases of the wreath product of wheels.* International Journal of Mathematical combinatorics, Vol. 1 (2008), 52-62 (2008).

[8] A. Kaveh, Structural Mechanics, *Graph and Matrix Methods. Research Studies Press*, Exeter, UK, 1992.

[9] D.J.A. Welsh, *Kruskal's theorem for matroids*, Proc. Cambridge Phil, Soc., 64, 3-4 (1968).