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Generalization of best proximity points theorem for non-self proximal contractions of first kind

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Abstract
The primary objective of this paper is the study of the generalization of some results given by Basha (Numer. Funct. Anal. Optim. 31:569–576, 2010). We present a new theorem on the existence and uniqueness of best proximity points for proximal β-quasi-contractive mappings for non-self-mappings $S: M \to N$ and $T: N \to M$. Furthermore, as a consequence, we give a new result on the existence and uniqueness of a common fixed point of two self mappings.

MSC: 47H10; 54H25

Keywords: Best proximity points; Proximal β-quasi-contractive mappings on metric spaces and proximal cyclic contraction

1 Introduction

In 1969, Fan in [2] proposed the concept best proximity point result for non-self continuous mappings $T: A \longrightarrow X$ where A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space X. He showed that there exists a such that $d(a, Ta) = d(Ta, A)$. Many extensions of Fan's theorems were established in the literature, such as in work by Reich [3], Sehgal and Singh [4] and Prolla [5].

In 2010, [1], Basha introduce the concept of best proximity point of a non-self mapping. Furthermore he introduced an extension of the Banach contraction principle by a best proximity theorem. Later on, several best proximity points results were derived (see e.g. [6–19]). Best proximity point theorems for non-self set valued mappings have been obtained in [20] by Jleli and Samet, in the context of proximal orbital completeness condition which is weaker than the compactness condition.

The aim of this article is to generalize the results of Basha [21] by introducing proximal β-quasi-contractive mappings which involve suitable comparison functions. As a consequence of our theorem, we obtain the result of Basha in [21] and an analogous result on proximal quasi-contractions is obtained which was first introduced by Jleli and Samet in [20].

2 Preliminaries and definitions

Let (M,N) be a pair of non-empty subsets of a metric space (X,d). The following notations will be used throughout this paper: $d(M,N) := \inf\{d(m,n) : m \in M, n \in N\}$; $d(x,N) := \inf\{d(x,n) : n \in N\}$.

Definition 2.1 ([1]) Let $T : M \to N$ be a non-self-mapping. An element $a_* \in M$ is said to be a best proximity point of T if $d(a_*, Ta_*) = d(M,N)$.

Note that in the case of self-mapping, a best proximal point is the normal fixed point, see [22, 23].

Definition 2.2 ([21]) Given non-self-mappings $S : M \to N$ and $T : N \to M$. The pair (S,T) is said to form a proximal cyclic contraction if there exists a non-negative number $k < 1$ such that

$$d(u,Sa) = d(M,N) \quad \text{and} \quad d(v,Tb) = d(M,N) \Longrightarrow d(u,v) \leq kd(a,b) + (1-k)d(M,N)$$

for all $u,a \in M$ and $v,b \in N$.

Definition 2.3 ([21]) A non-self-mapping $S : M \to N$ is said to be a proximal contraction of the first kind if there exists a non-negative number $\alpha < 1$ such that

$$d(u_1,Sa_1) = d(M,N) \quad \text{and} \quad d(u_2,Sa_2) = d(M,N) \Longrightarrow d(u_1,u_2) \leq \alpha d(a_1,a_2)$$

for all $u_1,u_2,a_1,a_2 \in M$.

Definition 2.4 ([24]) Let $\beta \in (0,+\infty)$. A β-comparison function is a map $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following properties:

(P_1) φ is nondecreasing.

(P_2) $\lim_{n\to\infty} \varphi_\beta^n(t) = 0$ for all $t > 0$, where φ_β^n denote the nth iteration of φ_β and $\varphi_\beta(t) = \varphi(\beta t)$.

(P_3) There exists $s \in (0,+\infty)$ such that $\sum_{n=1}^\infty \varphi_\beta^n(s) < \infty$.

(P_4) $(\mathrm{id} - \varphi_\beta) \circ \varphi_\beta(t) \leq \varphi_\beta \circ (\mathrm{id} - \varphi_\beta)(t)$ for all $t \geq 0$, where $\mathrm{id} : [0,\infty) \longrightarrow [0,\infty)$ is the identity function.

Throughout this work, the set of all functions φ satisfying (P_1), (P_2) and (P_3) will be denoted by Φ_β.

Remark 2.1 Let $\alpha,\beta \in (0,+\infty)$. If $\alpha < \beta$, then $\Phi_\beta \subset \Phi_\alpha$.

We recall the following useful lemma concerning the comparison functions Φ_β.

Lemma 2.1 ([24]) *Let* $\beta \in (0,+\infty)$ *and* $\varphi \in \Phi_\beta$. *Then*

(i) φ_β *is nondecreasing*;

(ii) $\varphi_\beta(t) < t$ *for all* $t > 0$;

(iii) $\sum_{n=1}^\infty \varphi_\beta^n(t) < \infty$ *for all* $t > 0$.

Definition 2.5 ([20]) A non-self-mapping $T: M \to N$ is said to be a proximal quasi-contraction if there exists a number $q \in [0,1)$ such that

$$d(u,v) \le q \max\{d(a,b), d(a,u), d(b,v), d(a,v), d(b,u)\}$$

whenever $a, b, u, v \in M$ satisfy the condition that $d(u, Ta) = d(M,N)$ and $d(v, Tb) = d(M,N)$.

3 Main results and theorems

Now, we start this section by introducing the following concept.

Definition 3.1 Let $\beta \in (0, +\infty)$. A non-self mapping $T: M \to N$ is said to be a proximal β-quasi-contraction if and only if there exist $\varphi \in \Phi_\beta$ and positive numbers $\alpha_0, \dots, \alpha_4$ such that

$$d(u,v) \le \varphi\big(\max\{\alpha_0 d(a,b), \alpha_1 d(a,u), \alpha_2 d(b,v), \alpha_3 d(a,v), \alpha_4 d(b,u)\}\big).$$

For all $a, b, u, v \in M$ satisfying, $d(u, Ta) = d(M,N)$ and $d(v, Tb) = d(M,N)$.

Let (M,N) be a pair of non-empty subsets of a metric space (X,d). The following notations will be used throughout this paper: $M_0 := \{u \in M:$ there exists $v \in N$ with $d(u,v) = d(M,N)\}$;$N_0 := \{v \in N:$ there exists $u \in M$ with $d(u,v) = d(M,N)\}$.

Our main result is giving by the following best proximity point theorems.

Theorem 3.1 *Let (M,N) be a pair of non-empty closed subsets of a complete metric space (X,d) such that M_0 and N_0 are non-empty. Let $S: M \longrightarrow N$ and $T: N \longrightarrow M$ be two mappings satisfying the following conditions*:

- (C_1) $S(M_0) \subset N_0$ *and* $T(N_0) \subset M_0$;
- (C_2) *there exist* $\beta_1, \beta_2 \ge \max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, 2\alpha_4\}$ *such that S is a proximal β_1-quasi-contraction mapping (say, $\psi \in \Phi_{\beta_1}$) and T is a proximal β_2-quasi-contraction mapping (say, $\phi \in \Phi_{\beta_2}$).*
- (C_3) *The pair (S,T) forms a proximal cyclic contraction.*
- (C_4) *Moreover, one of the following two assertions holds*:
  - (i) *ψ and ϕ are continuous*;
  - (ii) $\beta_1, \beta_2 > \max\{\alpha_2, \alpha_3\}$.

Then S has a unique best proximity point $a_* \in M$ and T has a unique best proximity point $b_* \in N$. Also these best proximity points satisfy $d(a_*, b_*) = d(M,N)$.

Proof Since M_0 is a non-empty set, M_0 contains at least one element, say $a_0 \in M_0$. Using the first hypothesis of the theorem, there exists $a_1 \in M_0$ such that $d(a_1, Sa_0) = d(M,N)$. Again, since $S(M_0) \subset N_0$, there exists $a_2 \in M_0$ such that $d(a_2, Sa_1) = d(M,N)$. Continuing this process in a similar fashion to find $a_{n+1} \in M_0$ such that $d(a_{n+1}, Sa_n) = d(M,N)$. Since S is a proximal β_1-quasi-contraction mapping for $\psi \in \Phi_{\beta_1}$ and since

$$d(a_{n+1}, Sa_n) = d(a_n, Sa_{n-1}) = d(M,N), \tag{1}$$

then by Definition 3.1 we have

$$\begin{aligned} d(a_{n+1},a_n) &\le \psi\bigl(\max\bigl\{\alpha_0 d(a_n,a_{n-1}),\alpha_1 d(a_n,a_{n+1}),\alpha_2 d(a_n,a_{n-1}),\alpha_4 d(a_{n+1},a_{n-1})\bigr\}\bigr) \\ &\le \psi\left(\max\left\{\begin{matrix}\alpha_0 d(a_n,a_{n-1}),\alpha_1 d(a_n,a_{n+1}),\alpha_2 d(a_n,a_{n-1}) \\ \alpha_4 d(a_{n-1},a_n)+\alpha_4 d(a_n,a_{n+1})\end{matrix}\right\}\right) \\ &\le \psi\left(\max\left\{\begin{matrix}\alpha_0 d(a_n,a_{n-1}),\alpha_1 d(a_n,a_{n+1}),\alpha_2 d(a_n,a_{n-1}) \\ 2\alpha_4 \max\{d(a_{n-1},a_n),d(a_n,a_{n+1})\}\end{matrix}\right\}\right) \\ &\le \psi\bigl(\beta_1 \max\bigl\{d(a_n,a_{n-1}),d(a_n,a_{n+1})\bigr\}\bigr) \\ &= \psi_{\beta_1}\bigl(\max\bigl\{d(a_n,a_{n-1}),d(a_n,a_{n+1})\bigr\}\bigr). \end{aligned} \tag{2}$$

Now, if $\max\{d(a_n,a_{n-1}),d(a_n,a_{n+1})\} = d(a_n,a_{n+1})$, then by Lemma 2.1 the above inequality becomes

$$d(a_{n+1},a_n) \le \psi_{\beta_1}\bigl(d(a_{n+1},a_n)\bigr) < d(a_{n+1},a_n),$$

which is a contradiction. Thus, $\max\{d(a_n,a_{n-1}),d(a_n,a_{n+1})\} = d(a_n,a_{n-1})$, then the above inequality (2) becomes

$$d(a_{n+1},a_n) \le \psi_{\beta_1}\bigl(d(a_{n-1},a_n)\bigr)).$$

By applying induction on n, the above inequality gives

$$d(a_{n+1},a_n) \le \psi^n_{\beta_1}\bigl(d(a_0,a_1)\bigr) \quad \forall n \ge 1. \tag{3}$$

Now, from the axioms of metric and Eq. (3), for positive integers $n < m$, we get

$$d(a_n,a_m) \le \sum_{k=n}^{m-1} d(a_k,a_{k+1}) \le \sum_{k=n}^{m-1} \psi^k_{\beta_1}\bigl(d(a_1,a_0)\bigr) \le \sum_{k=1}^{\infty} \psi^k_{\beta_1}\bigl(d(a_1,a_0)\bigr) < \infty.$$

Hence, for every $\epsilon > 0$ there exists $N > 0$ such that

$$d(a_n,a_m) \le \sum_{k=n}^{m-1} d(a_k,a_{k+1}) < \epsilon \quad \text{for all } m > n > N.$$

Therefore, $d(a_n,a_m) < \epsilon$ for all $m > n > N$. That is $\{a_n\}$ is a Cauchy sequence in M. But M is a closed subset of the complete metric space X, then $\{a_n\}$ converges to some element $a_* \in M$.

Since $T(N_0) \subset M_0$, by using a similar argument as above, there exists a sequence $\{b_n\} \subset N_0$ such that $d(b_{n+1},Tb_n) = d(M,N)$ for each n. Since T is a proximal β_2-quasi-contraction mapping (say $\phi \in \Phi_{\beta_2}$) and since $d(b_{n+1},Tb_n) = d(b_n,Tb_{n-1}) = d(M,N)$, we deduce from Definition 3.1 that

$$\begin{aligned} d(b_{n+1},b_n) &\le \phi\bigl(\max\bigl\{\alpha_0 d(b_n,b_{n-1}),\alpha_1 d(b_n,b_{n+1}),\alpha_2 d(b_n,b_{n-1}),\alpha_4 d(b_{n-1},b_{n+1})\bigr\}\bigr) \\ &\le \phi\left(\max\left\{\begin{matrix}\alpha_0 d(b_n,b_{n-1}),\alpha_1 d(b_n,b_{n+1}),\alpha_2 d(b_n,b_{n-1}), \\ \alpha_4 d(b_{n-1},b_n)+\alpha_4 d(b_n,b_{n+1})\end{matrix}\right\}\right) \end{aligned}$$

$$
\begin{aligned}
&\le \phi\left(\max\left\{\begin{matrix}\alpha_0 d(b_n,b_{n-1}),\alpha_1 d(b_n,b_{n+1}),\alpha_2 d(b_n,b_{n-1}),\\ 2\alpha_4\max\{d(b_{n-1},b_n),d(b_n,b_{n+1})\}\end{matrix}\right\}\right)\\
&\le \phi\bigl(\beta_2\max\bigl\{d(b_n,b_{n-1}),d(b_n,b_{n+1})\bigr\}\bigr)\\
&= \phi_{\beta_2}\bigl(\max\bigl\{d(b_n,b_{n-1}),d(b_n,b_{n+1})\bigr\}\bigr).
\end{aligned}
$$

Using a similar argument as in the case of $\{a_n\}$, one can show that $\{b_n\}$ is a Cauchy sequence in the closed subset N of the complete space X. Thus $\{b_n\}$ converges to $b_* \in N$. Now we shall show that a_* and b_* are best proximal points of S and T, respectively. As the pair (S,T) forms a proximal cyclic contraction, it follows that

$$d(a_{n+1},b_{n+1}) \le kd(a_n,b_n) + (1-k)d(M,N). \tag{4}$$

Taking the limit as $n \longrightarrow +\infty$, in Eq. (4) we get $d(a_*,b_*) \le kd(a_*,b_*) + (1-k)d(M,N)$, and so, $(1-k)d(a_*,b_*) \le (1-k)d(M,N)$. This implies

$$d(a_*,a_*) \le d(M,N). \tag{5}$$

Using the fact that $d(M,N) \le d(a_*,b_*)$ and (5), we get $d(a_*,b_*) = d(M,N)$. Therefore, we conclude that $a_* \in M_0$ and $b_* \in N_0$.

From one hand, since $S(M_0) \subset N_0$ and $T(N_0) \subset M_0$, there exist $u \in M$ and $v \in N$ such that

$$d(u,Sa_*) = d(v,Tb_*) = d(M,N). \tag{6}$$

On the other hand, by (1), (6) and using the hypothesis of the theorem that S is a proximal β_1-quasi-contraction mapping, we deduce that

$$
\begin{aligned}
&d(a_{n+1},u)\\
&\quad\le \psi\bigl(\max\bigl\{\alpha_0 d(a_n,a_*),\alpha_1 d(a_n,a_{n+1}),\alpha_2 d(a_*,u),\alpha_3 d(a_n,u),\alpha_4 d(a_*,a_{n+1})\bigr\}\bigr).
\end{aligned}
\tag{7}
$$

For simplicity, we denote

$$\rho = d(a_*,u)$$

and

$$A_n = \max\bigl\{\alpha_0 d(a_n,a_*),\alpha_1 d(a_n,a_{n+1}),\alpha_2 d(a_*,u),\alpha_3 d(a_n,u),\alpha_4 d(a_*,a_{n+1})\bigr\}.$$

Thus,

$$\lim_{n\longrightarrow+\infty} A_n = \max\{\alpha_2,\alpha_3\}\rho. \tag{8}$$

Now, we show by contradiction that $\rho = 0$. Suppose that $\rho > 0$. First, we consider the case where the assertion (i) of (C_4) is satisfied, that is, ψ is continuous. Then, taking the limit as $n \to \infty$ in (7) and using (8) and Lemma 2.1, we obtain

$$\rho \le \psi\bigl(\max\{\alpha_2,\alpha_3\}\rho\bigr) \le \psi(\beta_1\rho) = \psi_{\beta_1}(\rho) < \rho,$$

which is a contradiction. Now, we assume the case where the assertion (ii) of (C_4) is satisfied, that is, $\beta_1 > \max\{\alpha_2, \alpha_3\}$. Then there exist $\epsilon > 0$ and integer $N > 0$ such that, for all $n > N$, we have

$$A_n < (\max\{\alpha_2, \alpha_3\} + \epsilon)\rho \quad \text{and} \quad \beta_1 > \max\{\alpha_2, \alpha_3\} + \epsilon.$$

Therefore, the inequality (7) turns into the following inequality:

$$\begin{aligned} d(a_{n+1}, u) &\leq \psi(A_n) \\ &\leq \psi\big((\max\{\alpha_2, \alpha_3\} + \epsilon)\rho\big) = \psi_{\beta_1}\bigg(\frac{\max\{\alpha_2, \alpha_3\} + \epsilon}{\beta_1}\rho\bigg). \end{aligned}$$

Since $\psi \in \Phi_{\beta_1}$, by Lemma 2.1 we have

$$d(a_{n+1}, u) < \frac{\max\{\alpha_2, \alpha_3\} + \epsilon}{\beta_1}\rho < \rho.$$

By letting $n \to \infty$, the above inequality yields

$$\rho \leq \frac{\max\{\alpha_2, \alpha_3\} + \epsilon}{\beta_1}\rho < \rho,$$

which is a contradiction as well. Thus, in both two cases we get $0 = \rho = d(a_*, u)$, which means that $u = a_*$ and so from equation (6) we get $d(a_*, Sa_*) = d(M, N)$. That is a_* is a best proximity point for S.

Similarly, by using word by word the above argument after replacing u by v, S by T, β_1 by β_2 and ψ by ϕ, we get that $v = b_*$ and hence by (6) b_* is a best proximity point for the non-self mapping T.

Now, we shall prove that the obtained best proximity points a_* of S is unique. Assume to the contrary that there exists $x \in M$ such that $d(x, Sx) = d(M, N)$ and $x \neq a_*$. Since S is a proximal β_1-quasi-contractive mapping, we obtain

$$\begin{aligned} d(a_*, x) &\leq \psi\big(\max\big\{\alpha_0 d(a_*, x), \alpha_1 d(x, x), \alpha_2 d(a_*, a_*), \alpha_3 d(a_*, x), \alpha_4 d(a_*, x)\big\}\big) \\ &\leq \psi\big(\max\{\alpha_0, \alpha_3, \alpha_4\} d(a_*, x)\big) \\ &\leq \psi\big(\beta_1 d(a_*, x)\big) = \psi_{\beta_1}\big(d(a_*, x)\big) \\ &< d(a_*, x), \end{aligned}$$

which is a contradiction. Similarly, using the same as above and the fact that T is a proximal β_2-quasi-contractive mapping, we see that the best proximity point b_* of T is unique. □

In Theorem 3.1 by taking $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = 1, \beta_1 = \beta_2 = 1$ and $\psi(t) = \phi(t) = qt$ which is a continuous function and belongs to Φ_1, we obtain Corollary 3.3 in [21].

Corollary 3.1 *Let (M, N) be a pair of non-empty closed subsets of a complete metric space (X, d) such that M_0 and M_0 are non-empty. Let $S : M \longrightarrow N$ and $T : N \longrightarrow M$ be mappings satisfy the following conditions*:

(d_1) *$S(A_0) \subset M_0$ and $T(M_0) \subset N_0$.*
(d_2) *S and T are proximal quasi-contractions.*
(d_3) *The pair (S,T) form a proximal cyclic contraction.*

Then S has a unique best proximity point $a_* \in M$ such that $d(a_*, Sa_*) = d(M,N)$ and T has a unique best proximity point $b_* \in N$ such that $d(b_*, Tb_*) = d(M,N)$. Also, these best proximity points satisfies $d(a_*, b_*) = d(M,N)$.

Proof The result follows immediately from Theorem 3.1 by taking $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = \frac{1}{2}$, $\beta_1 = \beta_2 = 1$ and $\psi(t) = \phi(t) = qt$. □

The following definition, which was introduced in [24], is needed to derive a fixed point result as a consequence of our main theorem.

Definition 3.2 ([24]) Let X be a non-empty set. A mapping $T : X \longrightarrow X$ is called β-quasi-contractive, if there exist $\beta > 0$ and $\varphi \in \Phi_\beta$ such that

$$d(Ta, Tb) \leq \varphi\bigl(H_T(a,b)\bigr),$$

where

$$H_T(a,b) = \max\bigl\{\alpha_0 d(a,b), \alpha_1 d(a,Ta), \alpha_2 d(b,Tb), \alpha_3 d(a,Tb), \alpha_4 d(b,Ta)\bigr\},$$

with $\alpha_i \geq 0$ for $i = 0,1,2,3,4$.

Corollary 3.2 *Let (X,d) be a complete metric space. Let $S, T : X \longrightarrow X$ be two self-mappings satisfying the following conditions*:
(E_1) *S is β_1-quasi-contractive (say, $\psi \in \Phi_{\beta_1}$) and T is β_2-quasi-contractive (say, $\phi \in \Phi_{\beta_2}$).*
(E_2) *For all $a, b \in X$, $d(Sa, Tb) \leq kd(a,b)$ for some $k \in (0,1)$.*
(E_3) *Moreover, one of the following assertions holds*:
  (i) *ψ and ϕ are continuous*;
  (ii) $\beta_1, \beta_2 > \max\{\alpha_2, \alpha_3\}$.
Then S and T have a common unique fixed point.

Proof This result follows from Theorem 3.1 by taking $M = N = X$ and noticing that the hypotheses (E_1) and (E_2) of the corollary coincide with the first, second and the third conditions of Theorem 3.1. □

Example 3.1 Let $X = \mathbb{R}$ with the metric $d(x,y) = |x-y|$, then (X,d) is complete metric space. Let $M = [0,1]$ and $N = [2,3]$. Also, let $S : M \longrightarrow N$ and $T : N \longrightarrow M$ be defined by $S(x) = 3 - x$ and $T(y) = 3 - y$. Then it is easy to see that $d(M,N) = 1$, $M_0 = \{1\}$ and $N_0 = \{2\}$. Thus, $S(M_0) = S(\{1\}) = \{2\} = N_0$ and $T(M_0) = T(\{2\}) = \{1\} = M_0$.

Now we show that the pair (S,T) forms a proximal cyclic contraction. $d(u, Sa) = d(M,N) = 1$ implies that $u = a = 1 \in M$ and $d(v, Tb = d(M,N) = 1$ implies that $v = b = 2 \in N$.

Now, since $d(u,Sa) = d(1,S(1)) = d(1,2) = 1 = d(M,N)$ and $d(v,Tb) = d(2,T(2)) = d(2,1) = 1 = d(M,N)$. Therefore,

$$\begin{aligned} 1 = d(u,v) &= d(1,2) \\ &\le k\bigl(d(1,2)\bigr) + (1-k)d(M,N) \\ &= k + (1-k) = 1. \end{aligned}$$

So, (S,T) are proximal cyclic contraction for any $0 \le k < 1$. Now we shall show that S is proximal β_1-quasi-contraction mapping with $\psi(t) = \frac{1}{7}t, \beta_1 = 2$ and $\alpha_i = \frac{1}{5}$ for$i = 0,1,2,3$ and $\alpha_4 = \frac{1}{100}$. Note that $\psi(t) = \frac{1}{7}t \in \Phi_2$ since $\psi_{\beta_1}t = \psi_2 t = \frac{2}{7}t$. As above the only $a,b,u,v \in M$ such that $d(u,Sa) = d(M,N) = 1 = d(v,Sb)$ is $a = b = u = v = 1 \in M$. But

$$\begin{aligned} 0 = d(u,v) &= d(1,1) \\ &\le \frac{1}{7}\max\left\{\frac{1}{6}d(a,b), \frac{1}{6}d(a,u), \frac{1}{6}d(b,v), \frac{1}{6}d(a,v), \frac{1}{100}d(b,u)\right\} \\ &= \psi\left(\max\left\{\frac{1}{6}d(1,1), \frac{1}{100}d(1,1)\right\}\right) \\ &= \psi\bigl(\max\{0,0,0,0,0\}\bigr) \\ &= 0. \end{aligned}$$

So, S is a proximal β_1-quasi-contraction mapping. We deduce using our Theorem 3.1, that S has a unique best proximity point which is $a_* = 1$ in this example.

Similarly, by using the same argument as above, we can show that T is proximal β_2-quasi-contraction mapping with $\phi(t) = \frac{1}{8}t, \beta_2 = 3$ and $\alpha_i = \frac{1}{6}$ for$i = 0,1,2,3$ and $\alpha_4 = \frac{1}{100}$. Note that $\phi(t) = \frac{1}{8}t \in \Phi_3$ since $\phi_{\beta_2}t = \phi_3(t) = \frac{3}{8}t$. As above the only $a,b,u,v \in N$ such that $d(u,Ta) = d(M,N) = 1 = d(v,Tb)$ is $a = b = u = v = 2 \in M$. But

$$\begin{aligned} 0 = d(u,v) &= d((2,2) \\ &\le \frac{1}{8}\max\left\{\frac{1}{6}d(a,b), \frac{1}{6}d(a,u), \frac{1}{6}d(b,v), \frac{1}{6}d(a,v), \frac{1}{100}d(b,u)\right\} \\ &= \phi\left(\max\left\{\frac{1}{6}d(2,2), \frac{1}{100}d(2,2)\right\}\right) \\ &= \phi\bigl(\max\{0,0,0,0,0\}\bigr) \\ &= 0. \end{aligned}$$

So, T is a proximal β_2-quasi-contraction mapping. We deduce, using Theorem 3.1, that T has a unique best proximity point which is $b_* = 2$.

Finally, $\psi(t)$ and $\phi(t)$ are continuous mappings as well as $\beta_1, \beta_2 > \max_{0\le i\le 3}\{\alpha_i\}$. Therefore

$$d(a_*, b_*) = d(1,2) = 1 = d(M,N).$$

4 Conclusion

Improvements to some best proximity point theorems are proposed. In particular, the result due to Basha [21] for proximal contractions of first kind is generalized. Furthermore, we propose a similar result on existence and uniqueness of best proximity point of proximal quasi-contractions introduced by Jleli and Samet in [20]. This has been achieved by introducing β-quasi-contractions involving β-comparison functions introduced in [24].

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References

1. Basha, S.S.: Extensions of Banach's contraction principle. Numer. Funct. Anal. Optim. **31**, 569–576 (2010)
2. Fan, K.: Extension of two fixed point theorems of F.E Browder. Math. Z. **112**, 234–240 (1969)
3. Reich, S.: Approximate selections, best approximations, fixed points and invariant sets. J. Math. Anal. Appl. **62**, 104–113 (1978)
4. Sehgal, V.M., Singh, S.P.: A generalization to multifunctions of Fan's best approximation theorem. Proc. Am. Math. Soc. **102**, 534–537 (1988)
5. Prolla, J.B.: Fixed point theorems for set valued mappings and existence of best approximations. Numer. Funct. Anal. Optim. **5**, 449–455 (1983)
6. Basha, S.S.: Best proximity point theorems generalizing the contraction principle. J. Nonlinear Anal. Optim., Theory Appl. **74**, 5844–5850 (2011)
7. Basha, S.S.: Best proximity point theorems an exploration of a comon solution to approximation and optimization problems. Appl. Math. Comput. **218**, 9773–9780 (2012)
8. Basha, S.S., Sahrazad, N.: Best proximity point theorems for generalized proximal contractions. Fixed Point Theory Appl. **2012**, 42 (2012)
9. Sadiq Basha, S., Veeramani, P.: Best approximations and best proximity pairs. Acta Sci. Math. **63**, 289–300 (1997)
10. Sadiq Basha, S., Veeramani, P., Pai, D.V.: Best proximity pair theorems. Indian J. Pure Appl. Math. **32**, 1237–1246 (2001)
11. Sadiq Basha, S., Veeramani, P.: Best proximity pair theorems for multifunctions with open fibres. J. Approx. Theory **103**, 119–129 (2000)
12. Raj, V.S.: A best proximity point theorem for weakly contractive non-self-mappings. Nonlinear Anal. **74**, 4804–4808 (2011)
13. Karapinar, E.: Best proximity points of Kannan type-cyclic weak phi-contractions in ordered metric spaces. An. Ştiinţ. Univ. 'Ovidius' Constanţa **20**, 51–64 (2012)
14. Vetro, C.: Best proximity points: convergence and existence theorems for P-cyclic-mappings. Nonlinear Anal. **73**, 2283–2291 (2010)
15. Samet, B., Vetro, C., Vetro, P.: Fixed points theorems for α-ψ-contractive type mappings. Nonlinear Anal. **75**, 2154–2165 (2012)
16. Jleli, M., Karapinar, E., Samet, B.: Best proximity points for generalized α-ψ proximal contractives type mapping. J. Appl. Math. **2013**, Article ID 534127 (2013)
17. Aydi, H., Felhi, A.: On best proximity points for various α-proximal contractions on metric like spaces. J. Nonlinear Sci. Appl. **9**(8), 5202–5218 (2016)
18. Aydi, H., Felhia, A., Karapinar, E.: On common best proximity points for generalized α-ψ-proximal contractions. J. Nonlinear Sci. Appl. **9**(5), 2658–2670 (2016)

19. Ayari, M.I.: Best proximity point theorems for generalized α-β-proximal quasi-contractive mappings. Fixed Point Theory Appl. **2017**, 16 (2017)
20. Jleli, M., Samet, B.: An optimisation problem involving proximal quasi-contraction mapping. Fixed Point Theory Appl. **2014**, 141 (2014)
21. Sadiq, S.: Basha best proximity point theorems generalizing the contraction principle. Nonlinear Anal. **74**, 5844–5850 (2011)
22. Shatanawi, W., Mustafa, Z., Tahat, N.: Some coincidence point theorems for nonlinear contraction in ordered metric spaces. Fixed Point Theory Appl. **2011**, 68 (2011)
23. Shatanawi, W., Postolache, M., Mustafa, Z., Taha, N.: Some theorems for Boyd–Wong type contractions in ordered metric spaces. Abstr. Appl. Anal. **2012**, Article ID 359054 (2012). https://doi.org/10.1155/2012/359054
24. Ayari, M.I., Berzig, M., Kedim, I.: Coincidence and common fixed point results for β-quasi contractive mappings on metric spaces endowed with binary relation. Math. Sci. **10**(3), 105–114 (2016)