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DIAGNOSTIC CHECKING FOR LINEARITY IN TIME SERIES MODELS

BY

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ABSTRACT

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Title: Diagnostic checking for linearity in time series models

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In this thesis, I studied the well-known portmanteau tests appearing in the time series literature. In particular, I interest in reviewing the test statistics that can be used to check the adequacy of the fitted Autoregressive and Moving Average (ARMA) models, the Generalized Conditional Heteroskedasticity (GARCH) models, and special nonlinear models that are proposed early and widely used specially in financial time series. I estimate the empirical levels of these tests based on the Monte-Carlo significance tests and show that the Monte-Carlo tests provide an accurate estimate for these levels. I conduct a simulation power comparison between these tests and show that the Monte-Carlo significance test presented based on the determinant of a matrix which include four matrices of auto correlation of residual, auto correlation of squared residual and cross correlation between the residual and squared residuals has higher power than the other tests in many cases. I demonstrate the usefulness of the Monte-Carlo tests by applying these tests on the daily log-returns of Ooredoo Qatar.

Keywords: Portmanteau tests, ARMA model, GARCH model, Nonlinear model

DEDICATION

I dedicate my work and give many thanks to my wonderful family and friends. Particularly to my understanding and patient husband, son and parents for being by my side and encouraging me from the first day till now.

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CHAPTER 1: INTRODUCTION

1.1 Background

In the time-series statistical analysis, it is common to use the autoregressive-moving-average (ARMA) model and the autoregressive conditional heteroskedasticity (ARCH) to forecast the future values. The ARMA model is built on the assumption that the error terms are weak stationary and white noise. This means that, the errors (innovations or shocks) are homoscedastic and show no serial correlations. When the variance of the shocks is not a constant and depends on previous values of the process, the ARCH (or more generally GARCH) model can be used to estimate the volatility. The GARCH model stands for generalized autoregressive conditional heteroskedasticity where the volatility models are referred to as conditional heteroscedastic models.

After the identification and estimating of the parameters of a time series model, the diagnosis of the fitted model is the most important next step as suggested by the Box–Jenkins (1970) method. If the fitted model is adequate, the residuals should show no pattern and almost uncorrelated in time. Box and Pierce (1970) proposed to literature the portmanteau test under the assumptions of the ARMA model to check the validity of the assumptions and using the limiting distribution of the residual autocorrelation coefficients. Ljung and Box (1978) improved the Box-Pierce test by replacing the residual autocorrelation coefficients with their standardized values.

Monti (1994) proposed another portmanteau test based on the partial autocorrelations and showed that, the test is more powerful than initially proposed portmanteau test by Box and Pierce (1970) and Ljung and Box (1978), particularly when the fitted model underrate the order of the moving average component. The three tests of Box-Pierce, Ljung-Box, and Monti have the same asymptotic distribution as Chi-square. Peña and Rodríguez (2002) proposed a test based on the Toeplitz autocorrelation matrix. They considered their test based on the m^{th} root of the determinant of the m^{th} order of the autocorrelation matrix. They derived the limiting distribution of their test as a linear combination of Chi-squared distributions that was approximated by a Gamma distribution using the standardized values of residual autocorrelations. They showed that their test is more significant than the previous tests by Ljung and Box (1978) and Monti (1994) in many situations.

Peña and Rodríguez (2006) used the natural logarithm of the m plus one root of the determinant of the m^{th} Toeplitz autocorrelation matrix proposed by Peña and Rodríguez (2002). They derived the limiting distribution approximation of their test statistic as Gamma and showed that this test estimates the size more accurately than Peña and Rodríguez (2002).

One problem noticed by Lin and McLeod (2006) that is the size of the Peña and Rodríguez test statistics may not be significant based on the asymptotic approximation. To overcome this issue Lin and McLeod (2006) introduced the idea of using the Monte-Carlo significance test. They showed that the Monte-Carlo significance test provides a portmanteau test with the correct size with higher power than previous tests.

To check the linearity assumption in time series, many portmanteau tests have been proposed. Granger and Anderson (1978) and Maravall (1983) suggested testing for neglected nonlinearity in time series by looking at the autocorrelation function of the squared values of the time series. If the residuals are independent, then the squared residuals must be independent. If the model is nonlinear and the residuals are uncorrelated but not independent, then the plot of the autocorrelation function of residuals will show no serial correlation, whereas the plot of the autocorrelation function of squared residuals will indicate a serial correlation. In this regard, McLeod and Li (1983) introduced a portmanteau test to detect nonlinearity in time series models. Their test is essentially is the same as Ljung and Box (1978) where they squared the residuals in the autocorrelation coefficients instead of using directly the autocorrelation coefficients of the residuals.

Peña and Rodríguez (2002, 2006) showed that their tests may extend to test for nonlinear models including GARCH models by replacing autocorrelation of the residuals in the m^{th} order of the autocorrelation matrix by the autocorrelation of the squared residuals. They showed that their tests are more powerful in detecting the nonlinearity in time series than the McLeod and Li (1983) tests in many cases.

Rodríguez and Ruiz (2005) proposed a test for conditional heteroskedasticity takes into account not only the magnitude of the sample autocorrelations but also possible patterns among them. They noted the performance of their test with various alternative tests and showed that their test has more power than McLeod and Li (1983) and Peña and Rodríguez (2002) in many situations.

Mahdi and McLeod (2012) generalized Peña and Rodríguez (2002, 2006) and Lin and McLeod (2006) to the multivariate case. They found that the portmanteau test based on the Monte-Carlo significance test almost always outperforms the one based on the limiting distribution.

Recently, Psaradakis and Vávra (2019) proposed a portmanteau test for linearity of stationary time series using the generalized correlations of residuals from a linear model. The generalized correlations are the cross-correlations between different powers of the residuals *(r, s)* and autocorrelations for some natural numbers *r, s* which was introduced by Lawrance and Lewis (1985, 1987). Psaradakis and

Vávra (2019) applied their test for several linear and nonlinear models including ARMA and GARCH models and showed that their test is useful for testing ignored nonlinearity in time series models.

1.2 Thesis Layout

This thesis is divided into five chapters. The first chapter is the introduction, where we give a brief introduction about time series models and portmanteau test. The second chapter is a literature review for different portmanteau tests that have been published since 1970 till 2020, showing how the tests have been improved and updated throughout the years to be eligible to be used on various time series models. In the third chapter, we explained the Monte-Carlo significance portmanteau test proposed by Mahdi (2020) based on the autocorrelation and cross-correlation of the residuals and their squares. The fourth chapter includes the simulation study. The simulation study is divided into two parts: in section one type I error is estimated based on fitting AR (1), MA (1), ARCH and GARCH. Then a comparison is made between the tests. The second part of the chapter will compare the power of the tests based on the Monte-Carlo significance test as suggested by Lin and McLeod (2006). In the fifth chapter, we concentrate on the application part, where we use real financial data and check the adequacy of the fitted model based on the portmanteau statistics that we discuss in this thesis.

1.3 Research Objectives

1- To use the portmanteau test to check the linearity assumption in time series models.

2- We perform a simulation study comparing new test with popular portmanteau tests in time series literature.

3- We simulate data from ARMA/GARCH models.

4-We compare the performance of ARMA/GARCH models to get the best of the fitted model that can be used for forecasting.

5-We implement the advanced diagnostic checks in financial time series models on real data of Ooredoo returns.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

The process for modeling and analyzing the linear time series can be split into two parts of preprocessing and building the predictive model. In the preprocessing it requires to look through the raw data and calculate the asset return from the existing prices in the raw data, after that the stationary, autocorrelation, dynamic dependence of the time series should be checked. For analyzing time series, it is required that the time series satisfy at least weakly stationary. Where weak stationary means that the joint distribution of the return value of the asset over time should be time-invariant. Based on the pattern realized in the return value of a stationary time series, simple autoregressive model (AR), simple moving average model (MA), mixed autoregressive and moving average (ARMA), or models representing the seasonality in the data could be fitted, the appropriate L lag autocorrelation could be used for identifying the order of a moving average model and L lag partial autocorrelation is used for finding the proper lags for the autoregressive model. The extended autocorrelation function could be used for deriving the order for a simple ARMA model.

Mixing the AR and MA models is done in some cases to create a model that can adequately explain the dynamic structure of the data. The autoregressive and moving average ARMA *(p,q)* model has the following form:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \qquad (1)$$

After implementing AR or MA model, the residual should be checked to be normally distributed without serial correlation. There should be a constant variance in the residual. If the residual variance is not constant and has an increasing pattern over time, then the residual will have heteroskedasticity. In such cases, the error volatility could be modeled using the Autoregressive conditional heteroscedastic model. This methodology was developed by Engle (1982). If the residual shows a time-dependent pattern, then it could be modeled as two portions of time-dependent residual and a random term:

$$\varepsilon_t = \sigma_t z_t,$$

where ε_t is the innovation in time t, σ_t is time-dependent standard deviation and z_t is white noise. In the ARCH *(q)* model the time-dependent standard deviation part could be estimated by the following formula:

$$\sigma_t^2 = \theta_0 + \theta_1 \varepsilon_{t-1}^2 + \cdots + \theta_q \varepsilon_{t-q}^2, \qquad (2)$$

For evaluating the q in the ARCH model, the Lagrange multiplier test was suggested by Engle (1982). If the residual variance follows the AR model (it is serially correlated), then the ARCH method could be used for modeling the residual variance. Thus, in ARMA model one is modeling asset return and does predict the mean value of the asset return in a future period, whereas in the ARCH model, one is modeling the volatility of the asset return and will be able to predict the future variance of the asset return.

GARCH model was developed by Bollerslev (1986), The GARCH *(p,q)* model could be written as below:

$$\sigma_t^2 = \theta_0 + \theta_1 \varepsilon_{t-1}^2 + \cdots + \theta_q \varepsilon_{t-q}^2 + \phi_1 \sigma_{t-1}^2 + \cdots + \phi_p \sigma_{t-p}^2, \qquad (3)$$

Where, ε represents the error term at previous lags and σ represents the time-dependent standard deviation in previous lags. To test the existence of the GARCH *(p,q)* process, first, an AR*(p)* model will be fitted to the data. Then the correlation matrix for the residual will be calculated. This method is used when the residual include serial correlation and it has grown over time (heteroskedasticity) like the ARCH model. The difference is that in ARCH the residual follows the AR model. It means that partial autocorrelation function for residual show significant autocorrelation at some lags which suggests AR-pattern in the residual. In another way, using a Portmanteau test, the significance of serial correlation in the residual could be tested. If the residual follows the AR model, then the ARCH method is applicable. But if the residual follows the ARMA model, the methodology proposed by Bollerslev (1986) which suggests using a GARCH model is applicable. In literature, several types of GARCH models can be used for modeling the conditional heteroskedasticity. The GARCH-in-mean (GARCH-M) model proposed by Engle et al, (1987) adds a heteroskedasticity term into the mean equation that can be interpreted as a risk premium.The threshold GARCH model was developed by Zakoian (1994) is commonly used to handle leverage effects. This method suggests using a threshold for positive and negative error terms. The threshold was implemented by using dummy variables to separate the positive and negative coefficients of the error term in the GARCH formula. The formula of TGARCH (p,q) is as below:

$$\sigma_t = \theta_0 + \theta_1^+ \varepsilon_{t-1}^+ + \cdots + \theta_q^+ \varepsilon_{t-q}^+ + \cdots + \theta_1^- \varepsilon_{t-1}^- + \cdots + \theta_q^- \varepsilon_{t-q}^- + \phi_1 \sigma_{t-1} + \cdots + \phi_p \sigma_{t-p}, \quad (4)$$

Where ε_{t-1}^+ represents positive error terms and hence positive theta are coefficients for positive thresholds and ε_{t-1}^- represent negative error terms and the negative theta is coefficients for negative thresholds. The equation relates the standard deviation of the error term by previous standard deviation lags and previous error terms. This is essentially the same model proposed by Glosten et al, (1993). After TGARCH the Quadratic GARCH (QGARCH) model was proposed by Sentana (1995).

The exponential GARCH model (EGARCH) was proposed by Nelson (1991), the formula for this method is as below:

$$\varepsilon_t = \sigma_t z_t \,, \qquad ln\sigma_t^2 = \theta_0 + \sum_{i=1}^{p} \theta_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \phi_j ln\sigma_{t-j}^2, \tag{5}$$

This model differs from GARCH because it uses the log of variance instead of variance.

In 1970 Box-Jenkins introduced an approach called stochastic model building, this model is used to analyze and forecast time series models. A stochastic model is a process that is made up of three main stages:

1- Identification: detect the underlying pattern in the data using autocorrelation function and finding the most appropriate model using cross validation on the testing data or the model information criteria like Akaike information criteria (AIC) or Bayesian information criteria (BIC).
2- Estimation: fitting a proper model to the data using conditional sum of squares or the maximum likelihood estimation (MLE) and finding the parameter estimates for the model.
3- Diagnostic Checking: checking the assumptions of the model. The normality of the residuals. Whether the residual has constant variance and is homogenous or there are problems like non-constant variance or the residual is

heteroscedastic. Verifying that the residuals are independent from each other and there is no serial dependence in the residuals.

In my thesis, I concentrate on the third stage which the diagnostic checking using the portmanteau tests for linear and for nonlinear models.

2.2 Portmanteau Tests for Linear Models

Box and Pierce (1970) have analyzed the diagnostic of the model fit confirmation. They have written that "a proper model fit should lead to a residual which is like white noise (independent and identically distributed with mean zero and constant variance)". Thus, an appropriate model will have a residual which is not serially correlated and have constant variance. This means that there should be zero autocorrelation in the residual for an appropriate model fit. Box and Pierce (1970) have proposed the portmanteau test to check the residual autocorrelation, based on the null hypothesis that the autocorrelation of the residual at lag m (m is a natural number) equal to zero. Where the null hypothesis is $H_o : \rho_1 = \rho_2 = \dots = \rho_m = 0$ and the alternative hypothesis $H_a: \rho_i \neq 0$ for some $(i = 1, 2 \dots\dots, m)$. The proposed portmanteau statistic is given by:

$$Q^*(m) = T \sum_{l=1}^{m} \hat{\rho}_{1,1}^2(l), \qquad (6)$$

where T is the sample size and $\hat{\rho}_{1,1}(l)$ is the autocorrelation of the residuals at lag $(l = 1, 2 \dots\dots, m)$. They showed that the test $Q^*(m)$ is asymptotically distributed as Chi-squared distribution with m degrees of freedom. For an AR*(p)* process the degree of freedom will be $\nu = m - p$, for an MA*(q)* process the degree of freedom will be $\nu = m - q$ and for a mixed ARMA(p, q) process the degree of freedom of the Chi-squared distribution will be $\nu = m - p - q$. Hence, if the p-value of the portmanteau test is more than the significant level (usually 5%), then the test could not reject the null hypothesis at a 5% significance level and the assumption of no serial correlation is met. Box and pierce (1970) have tested the autocorrelation of the residual, for an AR process (autoregressive), MA process (moving average) and a mixed autoregressive moving average process (ARMA).

Ljung and Box (1978) have modified the portmanteau test proposed by Box and pierce (1970) to increase the power of the test. They presented some considerations about the power and robustness of the portmanteau test when the innovations are not normally distributed. The modified portmanteau test is given by:

$$Q(m) = T(T+2)\sum_{l=1}^{m} \frac{\hat{\rho}_{1,1}^{2}(l)}{T-l} \ , \tag{7}$$

They showed that the $Q(m)$ statistics has a asymptotic Chi-squared distribution with degree of freedom of $\nu = m - p - q$ for an ARMA *(p, q)* process. Ljung and Box (1978) have done Monte Carlo study to compare the power of their presented test with the previous portmanteau test and showed that the test $Q(m)$ was highly improved especially in small samples. In the calculation of $Q(m)$, there is more emphasis on the later autocorrelation compared with $Q^{*}(m)$. This is an advantage when the serial correlation occurs in the higher lags since the denominator takes value in lags $(T - l\,)$, so it could be seen that the weight of the autocorrelation of higher lags is more compared with lower lags in the calculation of $Q(m)$ statistic.

Monti (1994) proposed a portmanteau test statistic by using the residual partial autocorrelation. It was shown that the proposed test is more powerful when the fitted model of the ARMA process underestimates the order of the moving average component *(q)*. The performance of the test was checked by implementing using Monte Carlo experiment. The portmanteau test proposed by Monti (1994) has the same formula as Ljung and Box (1978), just the difference is that Monti uses residual partial autocorrelation instead. The presented test is given by:

$$\tilde{Q}(m) = T(T+2)\sum_{l=1}^{m}\frac{\hat{\pi}_{1,1}^{2}(l)}{T-l}\ , \tag{8}$$

where $\hat{\pi}_{1,1}(l)$ is the l^{th} lag partial autocorrelation of the residual. For a white noise residual, it is expected to have no partial autocorrelation in the residual, so the partial autocorrelation of the residual should not be significantly different from zero. The test statistics have asymptotically a Chi-squared distribution with $m - p - q$ degree of freedom in an ARMA *(p, q)* model.

Peña and Rodríguez (2002) proposed a portmanteau test based on the m^{th} order of autocorrelation matrix they showed that it is more powerful than the Ljung and Box (1978) and Monti (1994) test statistics. The proposed portmanteau test was tested on models with various sample sizes. Peña and Rodriguez (2002) have shown that the new test depending on sample size could improve and be more powerful than previous tests up to 50%. This test is capable of detecting nonlinearity in the residuals by replacing the autocorrelation of the residuals by the autocorrelation of the squared residuals. Researchers showed by examples and simulation that the test is, in general, more powerful than previously done test by McLeod and Li (1983). The presented test by Peña and Rodríguez (2002) is based on the determinant of the autocorrelation matrix. So in this test researchers first calculate the autocorrelation matrix for *m* lags and then the m^{th} root of the determinant for that autocorrelation matrix is calculated and is used in the formula for measuring the portmanteau test statistics. The researchers mentioned that this test could be seen as a linear combination of partial autocorrelations instead of autocorrelations. The proposed portmanteau test is as below:

$$\hat{D}_m = T\left(1 - \left|\hat{R}_m\right|^{\frac{1}{m}}\right), \tag{9}$$

where $\hat{R}_m$ is the m lag autocorrelation matrix of the residual for a stationary time series. T is the sample size and $\left|\hat{R}_m\right|^{\frac{1}{m}}$ is the m^{th} root of the determinant of the autocorrelation matrix that is given by:

$$\hat{R}_m = \begin{bmatrix} 1 & \hat{\rho}_{1,1}(1) & \cdots & \hat{\rho}_{1,1}(m) \\ \hat{\rho}_{1,1}(-1) & 1 & \cdots & \hat{\rho}_{1,1}(m-1) \\ \vdots & & \ddots & \vdots \\ \hat{\rho}_{1,1}(-m) & \hat{\rho}_{1,1}(1-m) & \ldots & 1 \end{bmatrix}, \tag{10}$$

The asymptotic distribution of this proposed Portmanteau test was shown to be a linear combination of the Chi-squared distribution. Peña and Rodríguez (2002) showed that the test statistics could be approximated as a Gamma distribution. The proposed statistics can be interpreted in two ways. The first one can be done by the utilization of recursive expression of the determinant of the correlation matrix $\hat{R}_m$. the expression $1 - \left|\hat{R}_m\right|^{\frac{1}{m}}$ is explained as the coefficient of average squared correlation.

The second method of interpretation of the proposed portmanteau statistic is by the coefficient of partial autocorrelation. The expression of the statistic using the second method is given below:

$$\widehat{D}_m = \left|\hat{R}_m\right|^{\frac{1}{m}} \approx \prod_{l=1}^{m} (1 - \hat{\pi}_{1,1}^{2}(l))^{(m+1-l)/m}, \qquad (11)$$

From the above relation, it is shown that the term $\left|\hat{R}_m\right|^{\frac{1}{m}}$ is the weighted function of initial m residual coefficients of partial autocorrelation. The significance level of the portmanteau test was studied in detail. In addition to this, the power of the new statistics was also judged and compared to the previous portmanteau test of Ljung and Box (1978) and Monti (1994). In the portmanteau test proposed by Monti (1994) and Ljung-Box (1978), the significance level and the power of the test statistics were measured using the percentiles distribution of the χ^2 while in this test this was done by a Gamma distribution. The significance level of the $\widehat{D}_m$ portmanteau statistics was tested using both, low order Autoregressive (AR) as well as Moving Average (MA) models. The test was conducted using 100 samples of observations with values of m (5, 10 and 20). The nominal levels were kept the same (1% & 5%) as they were used in the previous proposed test.

In most cases, it has been observed that the significance level of the Ljung-Box portmanteau test is greater than the corresponding nominal level. When the nominal level is 5 percent, the significance level of portmanteau test statistics lies in between the interval of (.041 to .055) while that of the Ljung-Box test lies in between the interval of (.052 to .069). From these results, it appears that the value of m does not affect the significance level of portmanteau statistics. To analyze the power of the test, 24 different models given by Monti (1994) are applied and in each case, it is found that the power of the test is inversely proportional to the value of m, so that increasing values of m result in a decrease of power of the portmanteau test. The performance of the test was also tested for a small sample of data. The performance test shows that before a decrease in power due to an increase in the value of m, the proposed statistics almost remain the most powerful and an increase in power can be up to 75 percent.

Peña and Rodríguez (2006) gave their modification of a portmanteau test for goodness of fit test in time series using the log of the determinant for the autocorrelation matrix. Their modified statistic is asymptotically equivalent to $\widehat{D}_m$ given in (11), but the modified test is 25 percent more powerful than their previous test especially in the case when the sample size is small. Two approximations are utilized for the test statistic. The first is normal and the other is Gamma approximation. The proposed portmanteau test proposed is given by:

$$D_m{}^* = -\frac{T}{m+1} log|\hat{R}_m| \,, \tag{12}$$

where $\hat{R}_m$ is the autocorrelation matrix of m lags given in *(10)*. After suitable modifications in the above expressions, the statistic in *(12)* can be written as:

$$D_m{}^* = -T \sum_{l=1}^{m} \frac{(m+1-l)}{m+1} log\left(1 - \hat{\pi}^2{}_{1,1}(l)\right), \tag{13}$$

The $D_m{}^*$statistic is proportional to squared partial autocorrelation coefficients weighted average in which the large weights are assigned to low order coefficients and smaller weights are assigned to large order coefficients. Where $\hat{\pi}^2{}_{1,1}$ is the partial autocorrelation between the residuals. The distribution of this statistic is asymptotically distributed as a sum of independent Chi-square that can be approximated by Gamma, but the performance has improved for the finite size of samples. Peña and Rodríguez (2006) showed that $D_m{}^*$can increase the power up to 50 percent than the test statistics of Ljung-Box, and Monti, depending upon the sample size and model.

Recently, Fisher and Gallagher (2012) proposed a weighted portmanteau test based on the trace of the square autocorrelation matrix and show that the asymptotic distribution is a sum of Chi-square that can be approximated as Gamma distribution. Their simulation study suggests that the weighted test is more powerful than Ljung-Box, Mahdi, and McLeod (2012) test in the ARMA process and it has easy computation and is numerically stable. The weighted Portmanteau test proposed by Fisher and Gallagher (2012) is presented below:

$$\tilde{Q}_W(m) = T(T+2)\sum_{l=1}^{m}\left(\frac{m-l+1}{m}\right)\frac{\hat{\rho}_{1,1}^2(l)}{T-l}, \tag{14}$$

This test is like the Ljung-Box test, but it is a weighted Ljung-Box test which gives the most weights to lag 1 and the lowest weight to lag m in the calculation of the portmanteau test statistics. This test also could be derived by using partial autocorrelation instead of the autocorrelation function and then will be called as a weighted Monti test statistic. The presented formula for the weighted Monti test which used partial autocorrelation is:

$$\widetilde{M}_W(m) = T(T+2)\sum_{l=1}^{m}\left(\frac{m-l+1}{m}\right)\frac{\hat{\pi}_{1,1}^2(l)}{T-l}, \qquad (15)$$

The formula is the same as the previous formula and just uses the partial autocorrelation of the residual (${\hat{\pi}^2}_{1,1}$) instead of autocorrelation.

2.3 Portmanteau Tests for Nonlinear Models

Sometimes the time series has a nonlinear pattern that requires implementing nonlinear analysis because the linear model is less flexible and has a bias in the computation. Testing the linearity assumption could be categorized into two groups. The first group is based on the Volterra expansion of stationary time series Wiener (1958). The formula of the Volterra expansion is given by:

$$r_t = \mu + \sum_{i=-\infty}^{\infty}\theta_i\varepsilon_{t-i} + \sum_{i,j=-\infty}^{\infty}\theta_{ij}\varepsilon_{t-i}\varepsilon_{t-j} + \sum_{i,j,k=-\infty}^{\infty}\theta_{ijk}\varepsilon_{t-i}\varepsilon_{t-j}\varepsilon_{t-k} + \cdots, \qquad (16)$$

Looking at the above formula, it could be seen that the time series is not only related to the linear term of the previous residual and not also linear related to the second and third-order of the residuals. The test for linearity assumption will be performed to see whether the higher-order coefficients are significantly different from zero or the null hypothesis is true, and the higher orders are set to zero. The above formula is for a strong stationary time series, where strong stationary is given when the distribution does not change over time instead it stays the same even when a shift in time is happened. μ is the mean of the series of (θ_{ijk}) are the regression parameters. If any of the higher-order coefficients of residuals (θ_{ij}), (θ_{ijk}) shown to be non-zero, then the time series will be nonlinear. Special cases of *(15)* will be introduced in the simulation chapter.

The linearity assumption is considered to be one of the most important assumptions in the ARMA model. When the time series is linear in the mean but not linear invariance, we will move to the ARCH model proposed by Engle et al. (1987) test that can be used to check the significance of the ARCH effect. For the second group of the tests, the linearity assumption will be tested by using the autocorrelation function on the higher order of time series (*squared value*). McLeod and Li (1983) proposed a test statistic to detect the nonlinearity in the time series, based on the autocorrelation of the squared residuals. McLeod and Li (1983) have used the idea that the autocorrelation of the squared residual is very useful in the diagnostic of the non-linear types of the serial dependence in the residual, for an ARMA process which shown by Granger and Anderson (1978). Their test statistic is given by

$$Q_{ML}^{*} = T(T+2)\sum_{l=1}^{m}\frac{\hat{\rho}_{2,2}^{2}(l)}{T-l}, \tag{17}$$

Where

$$\hat{\rho}_{22}(l) = \frac{\sum_{t=l+1}^{T}(\hat{\varepsilon}_t^2 - \hat{\sigma}^2)(\hat{\varepsilon}_{t-l}^2 - \hat{\sigma}^2)}{\sum_{t=1}^{T}(\varepsilon_t^2 - \hat{\sigma}^2)^2}, \tag{18}$$

McLeod and Li (1983) showed that the squared residual autocorrelation follows asymptotically a multivariate normal distribution with the unit covariance matrix. They have tested the validity of their proposed test in the small sample size.

They have done a review on the literature proposed for the portmanteau test and mentioned that Granger and Anderson (1978) have found instances of time series. That was modeled by Box and Jenkins (1976) which hadn't any dependence on the residual (the residual autocorrelation was not significant). But there was seen significant autocorrelation in the squared residual of the same time series. For this situation, Granger and Anderson (1978) were suggested that by fitting a bilinear model to the residual of the ARMA process the forecast result could be improved. McLeod and Li (1983) also found numerous time series in which although the residual of the best fitted ARMA model did not have any serial dependence and significant autocorrelation and although they met the model fit adequacy. But the squared residual of the ARMA model has significant autocorrelation. Therefore, the presented portmanteau test by McLeod and Li (1983) is the same with the Ljung and Box (1978) by this difference that instead of the autocorrelation of the residual they have used the autocorrelation of squared residual, with $\hat{\sigma}^2 = \sum_{t=1}^{T} \hat{\varepsilon}_t^2 / n$, is the variance of the squared residual. McLeod and Li (1983) have tested the performance of their statistic by small sample simulation using 21 models with 10,000 times for each one. It was shown that the performance of the Q_{ML}^{*} was almost consistent for low and larger sample size only in four models out of 21 models which had low sample size ($T = 50$) the Q_{ML}^{*} was slightly less than lower bound of the 95% confidence interval for the number of rejection of the null hypothesis. Fisher and Gallagher (2012) proposed a weighted test of McLeod-Li statistic and show that the asymptotic distribution of this statistic is Gamma where the power of the weighted test is almost always higher than McLeod-Li statistic in detecting nonlinearity. The weighted statistic is given by:

$$\tilde{Q}_W^*(m) = T(T+2)\sum_{l=1}^{m}\left(\frac{m-l+1}{m}\right)\frac{\hat{\rho}_{2,2}^2(l)}{T-l}, \tag{19}$$

Li and Mak (1994) noticed that the Box and Pierce test statistic cannot be approximated accurately to Chi-square distribution based on the squared residuals when the process has an ARCH structure. In this regard, Li and Mak (1994) proposed a modified test statistic under the assumptions of ARCH*(b)* model:

$$L(b,m) = T\sum_{k=b+1}^{m}\hat{\rho}_{2,2}^2(\,\hat{\varepsilon}_t^2(l)/\hat{h}_t)\,, \tag{20}$$

where $\hat{h}_t$ is the sample conditional variance. They estimate the asymptotic distribution of their test by Chi-square distribution with degrees of freedom $m-b$. Fisher and Colin (2012) proposed a weighted version of Li and Mak (1994). Their test statistic is given by

$$L_w(b,m) = T\sum_{k=b+1}^{m}\frac{(m-k+(b+1))}{m}\hat{\rho}_{2,2}^2(\,\hat{\varepsilon}_t^2(l)/\hat{h}_t)\,, \tag{21}$$

The portmanteau statistic of Fisher and Gallagher (2012) is found to be linearly combined Chi-squared random variables that can be approximated to a Gamma distribution. Fisher and Gallagher (2012) showed that the power of their test statistic is more than the other tests. This fact has been revealed by the results of the simulations done during the test. The test is more efficient especially, in the case of non-linear models 'detection which has a long memory.

Peña and Rodríguez (2002, 2006) extended their test statistics to test for nonlinear models including GARCH models by replacing autocorrelation of the residuals in the m^{th} order of the autocorrelation matrix by the autocorrelation of the squared residuals.

The two statistics, respectively, are

$$\widehat{D}_m(\hat{\varepsilon}_t^2) = T\left(\left|1 - \hat{R}_m(\hat{\varepsilon}_t^2)\right|^{\frac{1}{m}}\right), \tag{22}$$

And

$$D_m{}^*(\hat{\varepsilon}_t^2) = -\frac{T}{m+1} log\left|\hat{R}_m(\hat{\varepsilon}_t^2)\right|, \tag{23}$$

where $\hat{R}_m(\hat{\varepsilon}_t^2)$ is the autocorrelation matrix of the squared residuals. The asymptotic distribution of the two test statistics is estimated by Gamma. They showed that their tests are more powerful in detecting the nonlinearity in time series than the McLeod and Li (1983) tests in many cases.

Rodriguez and Ruiz (2005) proposed another portmanteau test statistic which is a powerful and improved version of previous tests. The test was proposed for the analysis of financial time series having high consistent volatilities. Rodriguez and Ruiz (2005) extended the $\widehat{D}_m$ and proposed a modification in that proposed statistic by introducing the logarithm of the determinant. Although the proposed test of squared residuals by Peña and Rodríguez (2002) was more powerful than the test statistics of McLeod and Li (1983), this test bears some more unique properties especially for large values of m making the test attractive and more powerful. The improved statistic of the portmanteau test proposed by Rodriguez and Ruiz (2005) is given below:

$$Q_i^{*}(M) = T\sum_{K=1}^{M-i}\left[\sum_{l=0}^{i}\hat{\rho}(k+l)\right]^2 \quad i = 0, \dots, M-1, \tag{24}$$

where $\hat{\rho}(k+l)$ denotes the standard samples of autocorrelation and $(k+l)$ denotes the order of these samples. The asymptotic distribution of the test in *(24)* has been approximated by two distributions and they are Normal and Gamma approximations. The test is powerful and possesses a unique property of providing two useful information. The test not only checks the deviation samples of autocorrelations with zero but also gives information regarding the patterns of coefficients $\rho(k)$ that can be possibly made from these samples. The power of their portmanteau test was tested concerning short memory models and long memory models. Artificial series was generated through the Autoregressive stochastic volatility model for finite samples to investigate the power of test statistic $Q_i^{*}(M)$. Comparing the powers of their test statistic with other previous tests, for a sample of size 100, suggested that the $Q_i^{*}(M)$ is higher than the other previous tests.

Lin and McLeod (2006) showed that most often the asymptotic distribution of the test statistic of Peña and Rodríguez does not correspond very well to Gamma approximation. The lack of correspondence occurs in the case of a small number of lags used in the test. The researchers suggested using the non-parametric Monte Carlo significance test of Peña and Rodríguez (2002). They showed that for a series of lengths less than 1000, the asymptotic distribution of the statistics of this test may be very slow therefore Monte-Carlo test is recommended owing to the fact of its higher power as compared to the test of Peña and Rodríguez and Ljung-Box portmanteau statistics.

The test statistics of these tests are considered to correspond well to autoregressive conditional heteroskedasticity (ARCH) models but they show some power lacking in other non-linear models that do not have obvious autoregressive conditional heteroskedasticity (ARCH) structures. In this regard, in addition to the test for cross-correlations between residuals and their squares, Psaradakis and Vavra (2019) proposed a new portmanteau test. Their test is used for checking the adequacy of fitted stationary time series using the generalized correlations of residuals from a linear model. They applied their test to several linear and nonlinear models including ARMA and GARCH models and showed that their test is useful for detecting ignored nonlinearity in time series models. The test statistics proposed for testing the linearity is given below:

$$Q_{rs}(m) = T \sum_{l=1}^{m} \hat{\rho}^2{}_{rs}\,(l) \;, \qquad (25)$$

And

$$\bar{Q}_{rs}(m) = T(T+2) \sum_{l=1}^{m} (T-l)^{-1}\, \hat{\rho}^2{}_{rs}(l)\,, \qquad (26)$$

where $r, s,$ and m are some natural numbers such that the sum of r and $s < 2$ and m is less than T. The asymptotic distributions of these tests are Chi-square. They claimed that their tests are more powerful than McLeod and Li (1983) in many cases.

Recently, Mahdi (2020) proposed a new that can be used simultaneously for both ARMA and ARCH/GARCH models. He proposed a new autocorrelation and cross-correlation test given by:

$$C_m = -\frac{T}{m+1} log \left|\hat{R}(m)\right| , \tag{27}$$

where $\left|\hat{R}(m)\right|$ is the determinant of the block matrix with a dimension of $2(m+1) \times 2(m+1)$ which is given by:

$$\hat{R}(m) = \begin{bmatrix} \hat{R}_{11}(m) & \hat{R}_{12}(m) \\ \hat{R}'_{12}(m) & \hat{R}_{22}(m) \end{bmatrix}, \tag{28}$$

where $'$ stands for the transpose of the matrix and

$$\hat{R}_{ij}(m) = \begin{bmatrix} \hat{\rho}_{ij}(0) & \hat{\rho}_{ij}(1) & \hat{\rho}_{ij}(3) & .. & \hat{\rho}_{ij}(m) \\ \hat{\rho}_{ij}(-1) & \hat{\rho}_{ij}(0) & \hat{\rho}_{ij}(1) & .. & \hat{\rho}_{ij}(m-1) \\ \vdots & .. & \ddots & \vdots & \vdots \\ \hat{\rho}_{ij}(-m) & \hat{\rho}_{ij}\big(-(m-1)\big) & .. & \hat{\rho}_{ij}(-1) & \hat{\rho}_{ij}(0) \end{bmatrix}, \quad i, j = 1,2 , \tag{29}$$

Thus, the extended version of $\left|\hat{R}(m)\right|$ is

$$|\hat{R}(m)| = \left|\begin{array}{cccc|cccc} 1 & \hat{\rho}_{11}(1) & \dots & \hat{\rho}_{11}(m) & \hat{\rho}_{12}(0) & \hat{\rho}_{11}(1) & \dots & \hat{\rho}_{12}(m) \\ \hat{\rho}_{11}(1) & 1 & \dots & \hat{\rho}_{11}(m-1) & \hat{\rho}_{12}(-1) & \hat{\rho}_{12}(0) & \dots & \hat{\rho}_{12}(m-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \hat{\rho}_{11}(-m) & \hat{\rho}_{11}(1-m) & \dots & 1 & \hat{\rho}_{12}(-m) & \hat{\rho}_{12}(1-m) & \dots & \hat{\rho}_{12}(0) \\ \hline \hat{\rho}_{21}(0) & \hat{\rho}_{21}(1) & \dots & \hat{\rho}_{21}(m) & 1 & \hat{\rho}_{21}(1) & \dots & \hat{\rho}_{22}(m) \\ \hat{\rho}_{21}(-1) & \hat{\rho}_{21}(0) & \dots & \hat{\rho}_{21}(m-1) & \hat{\rho}_{22}(-1) & 1 & \dots & \hat{\rho}_{22}(m-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \hat{\rho}_{21}(-m) & \hat{\rho}_{21}(1-m) & \dots & \hat{\rho}_{21}(0) & \hat{\rho}_{22}(-m) & \hat{\rho}_{22}(1-m) & \dots & 1 \end{array}\right|, \quad (30)$$

Here, $\hat{\rho}_{11}(m)$ is the autocorrelation coefficient between residuals, $\hat{\rho}_{22}(m)$ is the autocorrelation coefficient between the squared-residuals, and $\hat{\rho}_{12}(m)$ is the cross-correlation between the residuals and their squares values at lag time m that is defined as follows:

$$\hat{\rho}_{12}(m) = \frac{\hat{\gamma}_{12}(m)}{\sqrt{\hat{\gamma}_{11}(0)}\sqrt{\hat{\gamma}_{22}(0)}}, \quad (31)$$

where

$$\hat{\gamma}_{12}(m) = Cov(\hat{\varepsilon}_t, \hat{\varepsilon}_{t+m}^2) = \frac{1}{n}\sum_{l=1}^{n-m}\left(f(\varepsilon_t) - \overline{f(\varepsilon_t)}\right)\left(f(\varepsilon_{t+m}) - \overline{f(\varepsilon_{t+m})}\right), \quad (32)$$

Under the null hypothesis that the fitted model is accurate, we expect that the sample autocorrelations and the sample cross-correlations at different lag times to be very close to zero. Thus, for small values of $|\hat{R}(m)|$ that are close to zero, the null hypothesis will be rejected and the model is not good (linearity assumption is not valid). On the other hand, for large values that are close to one, the fitted model is good.

CHAPTER 3: COMPUTATION STUDY

In the field of a portmanteau test, many researchers have proposed different methods of calculating the portmanteau value for different time series models. Some have worked with ARMA and others worked on ARCH and GARCH. Recently, Mahdi (2020) proposed a new that can be used simultaneously for both ARMA and ARCH/GARCH models. He proposed a new autocorrelation and cross-correlation test.

In this regard, Mahdi (2020) derived the asymptotic distribution of the test in (27*)* as a linear combination of Chi-square random variables and approximates it to a Gamma distribution. One limitation that the asymptotic distribution can distort the empirical size and the Monte-Carlo significance test is recommended in such a case. The Monte-Carlo significance test is recommended by Lin and McLeod (2006) and Mahdi and McLeod (2012) and can be done by following steps:

1. Generate data from the ARMA-GARCH model or other nonlinear models.
2. Fit a time series model and find the residuals of this fitted model.
3. Apply the portmanteau test on the residuals and get the value of the test statistic. I call this value as an observed value.
4. Simulate data from the fitted model in step 2,
5. Fit a time series model to this simulated data,
6. Get the residuals of this fitted model,
7. Apply the portmanteau test on the residuals and get the value of the test statistic. I call this value as a calculated value,
8. Compare the observed value with the calculated value,
9. Replicate the steps I-V, 1000 times and count the average that the calculated value is greater than or equal to the observed value. The result of these steps

will give us the p-value of the portmanteau test for fitting the model in Step 2 to the simulated model in Step 1.

10. Replicate steps 1-3, 500 times and compute the estimated p-value based on the rejection frequencies.

In total there will be 1000 simulation and each simulation include 500 replications. The total number of iterations in this study will be 500000 iterations.

In the simulation study, we use the technique of the Monte-Carlo significance test to estimate the observed significance levels (type I error probability) of the test statistics$C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q(m), \tilde{Q}_W(m), {D_m}^*, Q_{ML}^*, \tilde{Q}_W * (m),$ and ${D_m}^*(\hat{\varepsilon}_t^2)$ given in *(27), (25), (25), (26), (26), (7), (14), (13), (17), (19),* and *(23)* respectively. The tests $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}$ are essentially checked the adequacy of the fitted model based on the cross-correlations between the residuals and their squares values, $\hat{\rho}_{12}$, the tests $Q(m), \tilde{Q}_W(m), {D_m}^*$ check the adequacy of the fitted model based on the autocorrelations of the residuals, and the tests $Q_{ML}^*, \tilde{Q}_W * (m)$, ${D_m}^*(\hat{\varepsilon}_t^2)$ check the adequacy of the fitted model based on the autocorrelations of the squared-residuals. We used the nominal levels of $\alpha = 0.01, 0.05,$ and $\alpha = 0.10$. After that, we study the power of the tests (1- type II error) by comparing the performance of these portmanteau tests. We conduct this simulation study by using different sample sizes for different linear and nonlinear models. In my simulation study, we have used some trusted R packages published in well-reputed journals: **forecast**, **tseries**, **portes**, **TSA**, **rugarch**, **fGarch** packages.

3.1. Significance Level

In this section we have presented the results found by fitting a model to a data generated by some ARMA and GARCH process. Table 1 shows the estimated significant levels correspond to nominal levels 0.01, 0.05, and 0.10, based on the test statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q(m), \tilde{Q}_W(m), D_m{}^*, Q^*_{ML}, \tilde{Q}_W * (m)$, and $D_m{}^*(\hat{\varepsilon}_t^2)$, when a true AR(1) model is fitted to a series of length $T = 100$ generated by AR(1) process with parameters $\emptyset_1 = 0.1, 0.3, 0.6$, and 0.9 at lags $m = 10$ and 20. As seen from the table results, all empirical significant levels are estimated very well and have close values to their nominal levels. We also estimate the significant level by considering the case of fitting a true MA*(1)* model to a series length $T = 100$ generated by MA*(1)* process with parameters $\theta_1 = 0.1, 0.3, 0.6$, and *0.9*. The results are shown in *Table 2* and suggest that the use of the Monte-Carlo version of the portmanteau test accurately estimates the size of the test. Besides, we checked the performance of the Monte-Carlo significance test in the case of ARMA-GARCH models at lags $m = 6$ and *12*. We generated data of different sample sizes 100, 300, 500, and 1000 from four different models and then we fit the true model. After that, we calculate the empirical level for the nominal level 0.05 by considering the Monte-Carlo significance of the tests $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q^*_{ML}, \tilde{Q}_W, \tilde{Q}_W * (m), D_m{}^*(\hat{\varepsilon}_t^2)$, $L(b, m)$ and $L_w(b, m)$**,** where $L(b, m)$ and $L_w(b, m)$ are given in *(20)* and *(21)*.

It is worth to note that we did not consider the tests $Q(m), \tilde{Q}_W(m)$, and $D_m{}^*$ as these tests are not designed to work with GARCH models. Tables 3 we use the four models are taken from the literature Psaradakis and Vavra (2019) that is widely used in financial time series:

Model 1: ARCH (1)

$$\varepsilon_t = \sigma_t z_t, \textit{ where } \sigma_t^2 = 0.1 + 0.6\varepsilon_{t-1}^2 ,$$

Model 2: ARCH (2)

$$\varepsilon_t = \sigma_t z_t, \textit{ where } \sigma_t^2 = 0.2 + 0.2\varepsilon_{t-1}^2 + 0.2\varepsilon_{t-2}^2 , \tag{33}$$

Model 3: GARCH (1,1)

$$\varepsilon_t = \sigma_t z_t, \textit{ where } \sigma_t^2 = 1 + 0.05\varepsilon_{t-1}^2 + 0.9\sigma_{t-1}^2 ,$$

Model 4: AR (1)-ARCH (1)

$$r_t = 0.2r_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \qquad \textit{where} \quad \sigma_t^2 = 0.2 + 0.2\varepsilon_{t-1}^2 ,$$

The results indicate that the Monte-Carlo technique is applicable to check the significance level correctly for the new test and for the previous proposed tests and the results indicates that for the four models the tests gives a result near to the nominal value in most of the cases.

Table 1. Empirical size at 1%, 5% and 10% for AR (1) model fitted as AR (1) for T = 100.

m	$\emptyset_1$	Based on $\hat{\rho}_{12}$					Based on $\hat{\rho}_{11}$			Based on $\hat{\rho}_{22}$		
		C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	$Q(m)$	$\tilde{Q}_W(m)$	$D_m{}^*$	Q^*_{ML}	$\tilde{Q}_W*(m)$	$D_m{}^*(\hat{\varepsilon}_t^2)$
							$\alpha = 0.01$					
10	0.1	0.014	0.006	0.006	0.011	0.012	0.015	0.016	0.014	0.009	0.003	0.015
	0.3	0.013	0.005	0.004	0.012	0.009	0.016	0.014	0.018	0.012	0.004	0.013
	0.6	0.010	0.007	0.002	0.009	0.012	0.013	0.015	0.011	0.015	0.008	0.011
	0.9	0.009	0.003	0.003	0.010	0.011	0.014	0.011	0.008	0.019	0.011	0.007
20	0.1	0.013	0.005	0.004	0.011	0.011	0.015	0.015	0.013	0.007	0.002	0.015
	0.3	0.011	0.005	0.005	0.011	0.010	0.016	0.012	0.016	0.011	0.002	0.014
	0.6	0.010	0.006	0.001	0.010	0.012	0.014	0.014	0.010	0.014	0.007	0.008
	0.9	0.009	0.002	0.002	0.011	0.010	0.010	0.011	0.008	0.016	0.008	0.006
							$\alpha = 0.05$					
10	0.1	0.044	0.031	0.028	0.044	0.042	0.036	0.038	0.039	0.041	0.041	0.040
	0.3	0.042	0.023	0.027	0.046	0.040	0.039	0.037	0.039	0.042	0.042	0.041
	0.6	0.043	0.024	0.024	0.048	0.043	0.037	0.040	0.041	0.043	0.044	0.042
	0.9	0.045	0.032	0.020	0.041	0.050	0.040	0.039	0.043	0.040	0.044	0.041
20	0.1	0.042	0.030	0.025	0.043	0.041	0.035	0.039	0.037	0.040	0.040	0.038
	0.3	0.040	0.021	0.024	0.044	0.040	0.037	0.041	0.036	0.041	0.040	0.039
	0.6	0.043	0.022	0.021	0.046	0.042	0.036	0.045	0.040	0.043	0.043	0.040
	0.9	0.044	0.030	0.018	0.040	0.048	0.038	0.045	0.042	0.038	0.041	0.037
							$\alpha = 0.10$					
10	0.1	0.096	0.041	0.051	0.097	0.095	0.061	0.096	0.088	0.064	0.067	0.065
	0.3	0.093	0.058	0.052	0.097	0.095	0.067	0.098	0.087	0.065	0.076	0.064
	0.6	0.098	0.057	0.047	0.099	0.100	0.070	0.097	0.089	0.057	0.065	0.074
	0.9	0.101	0.050	0.059	0.102	0.099	0.073	0.098	0.083	0.069	0.057	0.056
20	0.1	0.093	0.040	0.047	0.096	0.094	0.060	0.094	0.087	0.061	0.064	0.065
	0.3	0.095	0.056	0.049	0.097	0.093	0.065	0.092	0.085	0.063	0.075	0.058
	0.6	0.096	0.054	0.044	0.096	0.099	0.069	0.096	0.086	0.055	0.064	0.071
	0.9	0.099	0.047	0.056	0.101	0.100	0.070	0.097	0.080	0.067	0.057	0.053

Table 2. Empirical size at 1%,5% and 10% for MA (1) model fitted as MA(1) for T = 100.

m	θ_1	Based on $\hat{\rho}_{12}$					Based on $\hat{\rho}_{11}$			Based on $\hat{\rho}_{22}$		
		C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	$Q(m)$	$\tilde{Q}_W(m)$	${D_m}^*$	Q^*_{ML}	$\tilde{Q}_W * (m)$	${D_m}^*(\hat{\varepsilon}_t^2)$
$\alpha = 0.01$												
10	0.1	0.015	0.008	0.007	0.011	0.015	0.018	0.011	0.015	0.014	0.016	0.015
	0.3	0.016	0.014	0.016	0.010	0.011	0.016	0.015	0.014	0.013	0.016	0.014
	0.6	0.014	0.016	0.015	0.012	0.012	0.017	0.009	0.014	0.013	0.014	0.014
	0.9	0.012	0.014	0.013	0.009	0.012	0.014	0.012	0.013	0.012	0.013	0.013
20	0.1	0.014	0.007	0.005	0.010	0.012	0.017	0.011	0.015	0.013	0.017	0.014
	0.3	0.013	0.012	0.013	0.010	0.010	0.016	0.013	0.016	0.012	0.015	0.013
	0.6	0.011	0.015	0.013	0.011	0.011	0.016	0.007	0.014	0.012	0.012	0.014
	0.9	0.012	0.013	0.011	0.008	0.010	0.012	0.011	0.014	0.012	0.011	0.012
$\alpha = 0.05$												
10	0.1	0.044	0.033	0.045	0.045	0.046	0.039	0.039	0.042	0.039	0.043	0.041
	0.3	0.045	0.042	0.044	0.049	0.048	0.043	0.043	0.043	0.042	0.045	0.041
	0.6	0.043	0.036	0.035	0.046	0.047	0.035	0.040	0.041	0.040	0.042	0.039
	0.9	0.047	0.037	0.042	0.048	0.048	0.040	0.041	0.044	0.037	0.040	0.042
20	0.1	0.043	0.032	0.042	0.043	0.045	0.037	0.037	0.041	0.037	0.042	0.041
	0.3	0.045	0.041	0.044	0.048	0.046	0.041	0.041	0.041	0.040	0.043	0.040
	0.6	0.042	0.034	0.033	0.044	0.044	0.031	0.036	0.044	0.037	0.040	0.037
	0.9	0.046	0.036	0.038	0.046	0.047	0.037	0.037	0.048	0.044	0.038	0.040
$\alpha = 0.10$												
10	0.1	0.096	0.061	0.065	0.095	0.093	0.081	0.082	0.094	0.073	0.078	0.083
	0.3	0.095	0.059	0.067	0.095	0.096	0.082	0.083	0.095	0.064	0.076	0.076
	0.6	0.098	0.067	0.057	0.096	0.097	0.075	0.079	0.097	0.078	0.070	0.087
	0.9	0.104	0.063	0.058	0.098	0.102	0.078	0.087	0.098	0.072	0.078	0.069
20	0.1	0.092	0.060	0.065	0.093	0.091	0.080	0.080	0.094	0.072	0.074	0.082
	0.3	0.093	0.054	0.064	0.095	0.094	0.082	0.081	0.093	0.064	0.074	0.073
	0.6	0.097	0.061	0.054	0.095	0.095	0.074	0.075	0.098	0.074	0.068	0.084
	0.9	0.102	0.060	0.057	0.097	0.101	0.073	0.084	0.103	0.071	0.075	0.067

Table 3: Empirical size at 5% under GARCH models

m	T	Based on $\hat{\rho}_{12}$					Based on $\hat{\rho}_{22}$				
		C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	Q^{*}_{ML}	$\tilde{Q}_W * (m)$	$D_m{}^{*}(\hat{\varepsilon}_t^2)$	$L(\text{b},\text{m})$	$L_w(\text{b},\text{m})$
Model 1: ARCH(1) model is fitted to data from ARCH(1) process											
6	100	0.045	0.039	0.032	0.044	0.035	0.037	0.034	0.038	0.037	0.036
	300	0.047	0.039	0.036	0.046	0.036	0.035	0.037	0.040	0.038	0.038
	500	0.049	0.042	0.043	0.048	0.040	0.038	0.040	0.041	0.041	0.040
	1000	0.050	0.044	0.045	0.052	0.053	0.040	0.041	0.039	0.043	0.041
12	100	0.043	0.038	0.033	0.042	0.032	0.036	0.030	0.039	0.036	0.037
	300	0.045	0.037	0.034	0.044	0.035	0.035	0.038	0.037	0.035	0.037
	500	0.047	0.040	0.038	0.046	0.040	0.037	0.040	0.041	0.040	0.039
	1000	0.049	0.043	0.042	0.049	0.049	0.040	0.041	0.042	0.041	0.040
Model 2: ARCH(2) model is fitted to data from ARCH(2) process											
6	100	0.048	0.036	0.037	0.047	0.043	0.037	0.042	0.036	0.036	0.039
	300	0.047	0.038	0.038	0.046	0.046	0.040	0.040	0.038	0.039	0.038
	500	0.048	0.042	0.042	0.047	0.046	0.041	0.043	0.039	0.041	0.040
	1000	0.049	0.045	0.045	0.053	0.051	0.039	0.045	0.043	0.040	0.042
12	100	0.046	0.037	0.039	0.045	0.046	0.035	0.035	0.037	0.034	0.039
	300	0.048	0.039	0.041	0.045	0.045	0.038	0.038	0.036	0.039	0.037
	500	0.047	0.042	0.043	0.046	0.047	0.039	0.041	0.039	0.038	0.040
	1000	0.047	0.045	0.042	0.048	0.049	0.040	0.043	0.042	0.042	0.041
Model 3: GARCH(1,1) model is fitted to data from GARCH(1,1) process											
6	100	0.046	0.036	0.040	0.045	0.043	0.035	0.037	0.037	0.038	0.034
	300	0.047	0.039	0.041	0.046	0.045	0.034	0.039	0.039	0.035	0.037
	500	0.049	0.045	0.045	0.047	0.047	0.037	0.045	0.041	0.036	0.036
	1000	0.052	0.045	0.043	0.049	0.048	0.039	0.043	0.043	0.041	0.041
12	100	0.047	0.035	0.039	0.046	0.045	0.037	0.040	0.036	0.037	0.037
	300	0.049	0.037	0.044	0.048	0.048	0.036	0.038	0.035	0.033	0.036
	500	0.050	0.041	0.041	0.049	0.049	0.038	0.042	0.039	0.040	0.039
	1000	0.052	0.042	0.040	0.053	0.051	0.042	0.045	0.040	0.041	0.040
Model 4: AR(1)-ARCH(1) model is fitted to data from AR(1)-ARCH(1) process											
6	100	0.048	0.037	0.041	0.047	0.045	0.037	0.034	0.038	0.040	0.039
	300	0.048	0.036	0.039	0.045	0.045	0.039	0.037	0.037	0.039	0.038
	500	0.050	0.040	0.038	0.048	0.046	0.042	0.041	0.040	0.041	0.042
	1000	0.051	0.044	0.042	0.050	0.049	0.045	0.042	0.040	0.043	0.041
12	100	0.048	0.041	0.040	0.047	0.046	0.034	0.038	0.039	0.040	0.038
	300	0.047	0.043	0.041	0.046	0.045	0.036	0.040	0.039	0.038	0.039
	500	0.051	0.047	0.039	0.050	0.052	0.038	0.043	0.040	0.040	0.043
	1000	0.049	0.048	0.045	0.049	0.048	0.042	0.046	0.041	0.042	0.043

3.2. Power Study

In this section we conduct a similar study of Monti (1994) and Peña and Rodríguez (2002, 2006) in order to investigate the power of the test statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}$ $, Q(m), \tilde{Q}_W(m), D_m{}^*, Q^*_{ML}, \tilde{Q}_W * (m), D_m{}^*(\hat{\varepsilon}_t^2), L(b,m)$ and $L_w(b,m)$. Although, Monti (1994) and Peña and Rodríguez (2002, 2006) and other researchers investigate the power of the portmanteau tests when a false AR*(1)* model is fitted to a data generated by *12* ARMA*(2,2)* process as well as when a MA*(1)* model is fitted to a data generated by another *12* ARMA*(2,2)* process, we focus my attention to investigate the power of the portmanteau test statistics in the case of ARMA-GARCH models. In this regard, we generated my data from *24* ARMA*(2,2)*-ARCH*(1)* models, where the parameters of the ARMA*(2,2)* are the same parameters studied by Monti (1994) and Peña and Rodríguez (2002, 2006), whereas the parameters of the ARCH(1) are selected to be $(\theta_0, \theta_1) = (0.1, 0.6)$. After that, we calculated the empirical power of the test statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q(m), \tilde{Q}_W(m), D_m{}^*,\ Q^*_{ML}, \tilde{Q}_W * (m),$ and $D_m{}^*(\hat{\varepsilon}_t^2)$, based on the Monte-Carlo techniques, when a false AR *(1)* (or MA *(1)*) model is fitted to the generated data. we also check the power for detecting nonlinearity in the following eight nonlinear models taken from the model in (16):

$$Model\ 1:\ Y_t = \varepsilon_t - 0.4\varepsilon_{t-1} + 0.3\varepsilon_{t-2} + 0.5\varepsilon_t\varepsilon_{t-2}\ ,$$

$$Model\ 2:\ Y_t = \varepsilon_t - 0.3\varepsilon_{t-1} + 0.2\varepsilon_{t-2} + 0.4\varepsilon_{t-1}\varepsilon_{t-2} - 0.25\varepsilon_{t-2}^2\ , \qquad (34)$$

$$Model\ 3:\ Y_t = 0.4Y_{t-1} - 0.3Y_{t-2} + 0.5Y_{t-1}\varepsilon_{t-1} + \varepsilon_t\ ,$$

$$Model\ 4:\ Y_t = 0.4Y_{t-1} - 0.3Y_{t-2} + 0.5Y_{t-1}\varepsilon_{t-1} + 0.8\varepsilon_{t-1} + \varepsilon_t\ ,$$

$$Model\ 5:\ Y_t = 0.4Y_{t-1} - 0.3Y_{t-2} + (0.8 + 0.5Y_{t-1})\varepsilon_{t-1} + \varepsilon_t\ ,$$

$$Model\ 6:\ Y_t = 0.5 - (0.4 - 0.4\varepsilon_{t-1})Y_{t-1} + \varepsilon_t\ , \qquad (35)$$

$$Model\ 7:\ Y_t = 0.8\varepsilon_{t-2}^2 + \varepsilon_t\ ,$$

$Model\ 8$: $Y_t = \varepsilon_t + 0.3\varepsilon_{t-1} + (0.2 + 0.4\varepsilon_{t-1} - 0.25\varepsilon_{t-2})\varepsilon_{t-2}$,

The first four models *(34)* studied by Keenan (1985) and the other *(35)* were studied by Psaradakis and Vávra (2019). For these eight models, I estimate the empirical power of the statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q^*_{ML}, \tilde{Q}_W * (m),$ and $D_m{}^*(\hat{\varepsilon}_t^2)$ by fitting a false AR*(p)* model, where the order $1 \leq p \leq 4$ will be selected by the Akaike information criteria AIC (Akaike, 1974).

We finished this section by studying the power of the tests for detecting nonlinearity in some GARCH models that are commonly used in financial time series. In particular, we calculate the power the test statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q^*_{ML}, \tilde{Q}_W * (m), L(b,m)$ and $L_w(b,m)$, when a false ARCH*(1)* is fitted to data generated by ARCH*(2)*, ARCH*(3)*, and AR*(1)*-GARCH*(1,1)* models.

Table 4 shows the power of the statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q(m), \tilde{Q}_W(m),$ $D_m{}^*, Q^*_{ML}, \tilde{Q}_W * (m),$ and $D_m{}^*(\hat{\varepsilon}_t^2)$ corresponding to the significance level 0.05 at lags 10 and 20 when an AR *(1)* model is erroneous fits to a Gaussian series of length 100 generated by from 12 ARMA *(2,2)*-ARCH*(1)* models. The parameters of the ARMA*(2,2)* are the models from 1 to 12 taken from Monti (1994) and Peña and Rodríguez (2002, 2006), whereas the parameters of the ARCH(1), which are studied by Psaradakis and Vávra (2019), are selected to be $(\theta_0, \theta_1) = (0.1, 0.6)$. The results suggest that the Power of the test C_m is higher than the other tests in most cases. It can be seen from the results in Table 4 that the power of the test C_m corresponds to models 2, 5, 9, and 11 tends to be less than the powers of the tests $Q^*_{ML}, \tilde{Q}_W(m), D_m{}^*(\hat{\varepsilon}_t^2)$, where the test $\tilde{Q}_W(m)$ is more powerful than the other tests for the models 2, 5, and 9, whereas the statistic $D_m{}^*$ has the highest power in model 11.

Table 5 shows the power of the statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q(m), \tilde{Q}_W(m), D_m{}^*$ $, Q^*_{ML}, \tilde{Q}_W * (m),$ and $D_m{}^*(\hat{\varepsilon}_t^2)$ corresponding to the significance level 0.05 at lags 10 and

20 when an MA*(1)* model is erroneous fits to a Gaussian series of length 100 generated by from 12 ARMA*(2,2)*-ARCH*(1)* models. The parameters of the ARMA*(2,2)* are the models 13-24 studied by Monti (1994) and Peña and Rodríguez (2002, 2006), whereas the parameters of the ARCH(1), which are studied by Psaradakis and Vavra (2019), are selected to be $(\theta_0, \theta_1) = (0.1, 0.6)$. we noticed that the power of the test C_m is the highest except for the models 14, 15, 17, 21, and 24, where the test ${D_m}^*(\hat{\varepsilon}_t^2)$ tends to have the highest power.

Table *6* gives the power of the statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q_{ML}^*, \tilde{Q}_W * (m),$ and ${D_m}^*(\hat{\varepsilon}_t^2)$ when a false AR (p) model, where the$1 \leq p \leq 4$, is fitted to a sample of length 200 generated by the eight nonlinear models given by (34) and (35) consider the lags 7 and 12. Here, the order p is selected via the Akaike information criteria AIC (Akaike, 1974). I used the function auto.arima () available from the R package "**forecast**" to select the best model with the smallest AIC value. The power of the statistic C_m is much higher than the other tests suggesting that the C_m has substantially improved for detecting nonlinearity in time series.

Finally, Table 7 shows the power of the tests statistics $C_m, Q_{12}, Q_{21}, \bar{Q}_{12}, \bar{Q}_{21}, Q_{ML}^*, \tilde{Q}_W * (m), {D_m}^*(\hat{\varepsilon}_t^2), L(b,m)$ and $L_w(b,m)$ at significance level 0.05 and lags 6 and 12, when a false ARCH (1) model is fitted to a series of length 100, 300, and 500 are generated by three different AR-GARCH models. The first model is ARCH *(2)* model with parameters$(0.2, 0.2, 0.2)$. The second model is ARCH *(3)* with parameters $(0.2, 0.2, 0.2, 0.2)$ and the last one is an AR *(1)*-GARCH*(1,1)* model with parameters *(0.2, 1, 0.05, 0.90)*. As seen from this table, one can conclude that the Monte-Carlo significance test of the statistic C_m almost always gives the highest power comparing to the other tests.

Table 4. Power of the tests when data generated from ARMA(2,2)-ARCH(1)and AR(1) fitted

Model	ϕ_1	ϕ_2	θ_1	θ_2	Based on $\hat{\rho}_{12}$					Based on $\hat{\rho}_{11}$			Based on $\hat{\rho}_{22}$		
					C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	$Q(m)$	$\tilde{Q}_W(m)$	$D_m{}^*$	Q^*_{ML}	$\tilde{Q}_W*(m)$	$D_m{}^*(\hat{\varepsilon}_t^2)$
								Lag 10							
1	----	---	-0.5	---	**0.741**	0.256	0.158	0.298	0.312	0.012	0.169	0.125	0.658	0.711	0.754
2	---	---	-0.8	---	0.674	0.147	0.274	0.301	0.320	0.011	0.120	0.147	0.874	**0.894**	0.841
3	---	---	-0.6	0.30	**0.595**	0.127	0.098	0.374	0.410	0.147	0.305	0.039	0.555	0.541	0.500
4	0.1	0.30	---	---	**0.649**	0.197	0.035	0.084	0.090	0.269	0.224	0.210	0.612	0.619	0.589
5	1.3	-0.35	---	---	0.756	0.307	0.198	0.274	0.314	0.097	0.127	0.078	0.894	**0.971**	0.310
6	0.7	---	-0.4	---	**0.761**	0.123	0.325	0.355	0.333	0.147	0.123	0.169	0.478	0.347	0.674
7	0.7	---	-0.9	---	**0.572**	0.105	0.214	0.421	0.458	0.199	0.201	0.310	0.511	0.498	0.537
8	0.4	---	-0.6	0.3	**0.681**	0.175	0.099	0.441	0.384	0.232	0.238	0.156	0.632	0.674	0.567
9	0.7	---	0.7	-0.15	0.945	0.164	0.147	0.217	0.204	0.397	0.300	0.268	0.952	**0.981**	0.902
10	0.7	0.2	0.5	---	**0.798**	0.087	0.213	0.458	0.461	0.101	0.185	0.247	0.718	0.785	0.751
11	0.7	0.2	-0.5	---	0.753	0.151	0.247	0.574	0.074	0.223	0.157	0.081	0.832	0.859	**0.910**
12	0.9	-0.4	1.20	-0.30	**0.997**	0.214	0.222	0.468	0.374	0.486	0.119	0.121	0.912	0.974	0.934
								Lag 20							
1	----	---	-0.5	---	**0.704**	0.203	0.114	0.264	0.294	0.006	0.137	0.101	0.631	0.689	0.721
2	---	---	-0.8	---	0.651	0.111	0.254	0.279	0.299	0.005	0.099	0.113	0.843	**0.855**	0.811
3	---	---	-0.6	0.30	**0.543**	0.099	0.047	0.345	0.387	0.111	0.259	0.020	0.521	0.501	0.474
4	0.1	0.30	---	---	**0.592**	0.159	0.020	0.065	0.045	0.219	0.198	0.187	0.589	0.587	0.562
5	1.3	-0.35	---	---	0.702	0.287	0.158	0.255	0.283	0.047	0.107	0.041	0.854	**0.935**	0.287
6	0.7	---	-0.4	---	**0.705**	0.092	0.301	0.310	0.302	0.114	0.092	0.124	0.436	0.311	0.651
7	0.7	---	-0.9	---	**0.531**	0.085	0.198	0.397	0.401	0.155	0.157	0.287	0.487	0.451	0.511
8	0.4	---	-0.6	0.3	**0.631**	0.153	0.050	0.411	0.355	0.209	0.203	0.121	0.604	0.642	0.541
9	0.7	---	0.7	-0.15	0.908	0.133	0.114	0.196	0.187	0.364	0.281	0.234	0.935	**0.943**	0.897
10	0.7	0.2	0.5	---	**0.755**	0.063	0.198	0.424	0.426	0.087	0.150	0.216	0.674	0.732	0.724
11	0.7	0.2	-0.5	---	0.702	0.119	0.207	0.541	0.035	0.186	0.107	0.023	0.801	0.831	**0.899**
12	0.9	-0.4	1.20	-0.30	**0.957**	0.187	0.197	0.430	0.344	0.438	0.096	0.094	0.899	0.934	0.901

Table 5: Power of the tests when data generated from ARMA (2,2)-ARCH(1)and MA(1) fitted

Model	ϕ_1	ϕ_2	θ_1	θ_2	Based on $\hat{\rho}_{12}$					Based on $\hat{\rho}_{11}$			Based on $\hat{\rho}_{22}$		
					C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	$Q(m)$	$\tilde{Q}_W(m)$	$D_m{}^*$	Q_{ML}^*	$\tilde{Q}_W*(m)$	$D_m{}^*(\hat{\varepsilon}_t^2)$
Lag 10															
13	0.5	---	---	---	**0.919**	0.354	0.219	0.231	0.427	0.654	0.521	0.614	0.853	0.891	0.759
14	0.8	---	---	---	0.615	0.278	0.306	0.489	0.502	0.547	0.630	0.798	0.681	0.803	**0.841**
15	1.10	-0.35	---	---	0.581	0.397	0.289	0.352	0.419	0.484	0.514	0.811	0.874	0.740	**0.983**
16	----	---	0.8	-0.5	**0.820**	0.219	0.347	0.498	0.301	0.539	0.761	0.617	0.708	0.698	0.541
17	---	---	-0.6	0.3	0.532	0.238	0.141	0.316	0.487	0.697	0.478	0.570	0.413	0.814	**0.893**
18	0.50	---	-0.7	---	**0.819**	0.341	0.387	0.397	0.398	0.414	0.613	0.784	0.597	0.650	0.614
19	-0.50	---	0.7	---	**0.925**	0.174	0.474	0.314	0.147	0.358	0.877	0.562	0.669	0.841	0.726
20	0.30	---	0.8	-0.5	**0.823**	0.439	0.411	0.479	0.213	0.674	0.612	0.710	0.741	0.512	0.801
21	0.80	---	-0.5	0.3	0.436	0.310	0.463	0.147	0.318	0.479	0.687	0.661	0.587	0.496	**0.734**
22	1.20	-0.5	0.9	---	**0.809**	0.427	0.374	0.298	0.204	0.591	0.510	0.719	0.749	0.698	0.417
23	0.30	-0.2	-0.7	---	**0.996**	0.251	0.310	0.599	0.421	0.687	0.841	0.397	0.452	0.532	0.630
24	0.90	-0.4	1.20	-0.3	0.689	0.399	0.474	0.497	0.471	0.699	0.447	0.477	0.706	0.754	**0.996**
Lag 20															
13	0.5	---	---	---	**0.878**	0.322	0.195	0.197	0.364	0.601	0.487	0.587	0.801	0.853	0.623
14	0.8	---	---	---	0.594	0.339	0.289	0.453	0.480	0.513	0.605	0.742	0.654	0.729	**0.800**
15	1.10	-0.35	---	---	0.563	0.347	0.262	0.304	0.387	0.432	0.494	0.784	0.862	0.713	**0.952**
16	----	---	0.8	-0.5	**0.785**	0.294	0.301	0.436	0.276	0.510	0.729	0.601	0.693	0.649	0.522
17	---	---	-0.6	0.3	0.504	0.107	0.202	0.299	0.423	0.647	0.438	0.526	0.402	0.784	**0.864**
18	0.50	---	-0.7	---	**0.774**	0.300	0.330	0.351	0.341	0.353	0.574	0.747	0.576	0631	0.587
19	-0.50	---	0.7	---	**0.903**	0.128	0.433	0.217	0.120	0.307	0.831	0.507	0.631	0.811	0.692
20	0.30	---	0.8	-0.5	**0.798**	0.419	0.397	0.431	0.174	0.651	0.451	0.689	0.705	0.497	0.775
21	0.80	---	-0.5	0.3	0.401	0.280	0.421	0.102	0.299	0.458	0.649	0.623	0.546	0.450	**0.705**
22	1.20	-0.5	0.9	---	**0.797**	0.404	0.337	0.237	0.193	0.563	0.496	0.687	0.717	0.674	0.398
23	0.30	-0.2	-0.7	---	**0.984**	0.229	0.301	0.540	0.397	0.640	0.810	0.352	0.431	0.498	0.611
24	0.90	-0.4	1.20	-0.3	0.621	0.350	0.432	0.531	0.432	0.652	0.406	0.408	0.686	0.719	**0.934**

Table 6. Power of the test of fitting false AR(p) on eight nonlinear models

Model	Lag 7								Lag 14							
	C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	Q^*_{ML}	$\tilde{Q}_W * (m)$	$D_m{}^*(\hat{\varepsilon}_t^2)$	C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	Q^*_{ML}	$\tilde{Q}_W * (m)$	$D_m{}^*(\hat{\varepsilon}_t^2)$
1	**0.708**	0.170	0.241	0.361	0.497	0.684	0.697	0.571	**0.673**	0.159	0.210	0.323	0.442	0.652	0.654	0.521
2	**0.742**	0.364	0.374	0.544	0.301	0.541	0.456	0.498	**0.709**	0.321	0.321	0.518	0.284	0.510	0.418	0.432
3	**0.100**	0.674	0.447	0.855	0.347	0.716	0.741	0.398	**0.943**	0.614	0.408	0.814	0.326	0.684	0.700	0.364
4	**0.984**	0.431	0.458	0.413	0.578	0.621	0.657	0.479	**0.923**	0.401	0.419	0.398	0.521	0.603	0.611	0.421
5	**0.987**	0.145	0.217	0.413	0.356	0.347	0.310	0.401	**0.954**	0.123	0.184	0.362	0.313	0.307	0.299	0.389
6	**1.000**	0.654	0.574	1.000	0.741	0.147	0.521	0.784	**0.979**	0.608	0.512	0.988	0.721	0.103	0.497	0.762
7	**0.897**	0.441	0.703	0.544	0.630	0.368	0.987	0.600	**0.861**	0.409	0.689	0.511	0.610	0.327	0.943	0.589
8	0.520	0.147	0.129	0.639	0.497	0.478	**0.590**	0.578	0.483	0.104	0.103	0.605	0.441	0.439	**0.555**	0.543

Table 7: Power of the tests at 5% for four nonlinear models and fitted model is ARCH (1)

m	T	Based on $\hat{\rho}_{12}$					Based on $\hat{\rho}_{22}$				
		C_m	Q_{12}	Q_{21}	$\bar{Q}_{12}$	$\bar{Q}_{21}$	Q_{ML}^{*}	$\tilde{Q}_W * (m)$	$D_m{}^{*}(\hat{\varepsilon}_t^2)$	$L(\mathrm{b,m})$	$L_w(\mathrm{b,m})$
Model 1: ARCH(1) model is fitted to data from ARCH(2) process with parameters (0.2,0.2,0.2)											
6	100	0.344	0.096	0.040	0.147	0.170	0.341	0.471	**0.479**	0.321	0.429
	300	**0.426**	0.214	0.189	0.189	0.192	0.247	0.400	0.421	0.350	0.402
	500	**0.450**	0.198	0.174	0.236	0.269	0.311	0.321	0.425	0.420	0.378
12	100	0.323	0.072	0.021	0.120	0.147	0.310	0.444	**0.449**	0.300	0.389
	300	**0.410**	0.187	0.143	0.143	0.140	0.222	0.386	0.399	0.323	0.473
	500	**0.437**	0.157	0.132	0.204	0.209	0.287	0.302	0.400	0.402	0.348
Model 2: ARCH(1) model is fitted to data from ARCH(3) process with parameters (0.2,0.2,0.2, 0.2)											
6	100	**0.485**	0.201	0.214	0.127	0.184	0.297	0.347	0.398	0.441	0.471
	300	**0.587**	0.258	0.350	0.274	0.257	0.348	0.451	0.477	0.548	0.530
	500	0.609	0.177	0.210	0.238	0.279	0.471	0.574	0.558	**0.614**	0.601
12	100	**0.441**	0.174	0.187	0.101	0.142	0.257	0.322	0.342	0.411	0.450
	300	**0.537**	0.211	0.321	0.249	0.239	0.309	0.427	0.439	0.517	0.509
	500	0.574	0.147	0.198	0.207	0.244	0.423	0.531	0.527	**0.582**	0.564
Model 1: ARCH(1) model is fitted to data from AR(1)-GARCH(1,1) process with parameters (0.2, 1, 0.05, 0.90)											
6	100	**0.741**	0.319	0.247	0.301	0.341	0.581	0.739	0.689	0.674	0.698
	300	**0.846**	0.387	0.398	0.241	0.297	0.498	0.687	0.839	0.812	0.739
	500	0.970	0.410	0.420	0.355	0.399	0.502	0.874	0.964	0.955	**0.972**
12	100	**0.705**	0.287	0.223	0.283	0.311	0.542	0.701	0.641	0.641	0.662
	300	**0.819**	0.352	0.347	0.202	0.247	0.468	0.643	0.815	0.783	0.701
	500	0.950	0.374	0.392	0.304	0.363	0.479	0.837	0.925	0.913	**0.954**

CHAPTER 4: APPLICATION OF REAL FINANCIAL DATA

4.1 Application part 1

The financial time series field had faced new development in recent years, in stochastic volatility, high-frequency trading, and new software utilities. In financial time series, the valuation of the asset over time is taken into account. The behavior of the financial market could be realized empirically however statistical theories play an important role in making an inference in financial time series. Instead of direct analysis on price, in most cases, the return series is considered for the statistical analysis. Also, volatility is considered an important variable in option pricing and risk management. The volatility of the return series varies over time and it could be separated in certain clusters. The evolution of the conditional variance in the time series of return is required to be analyzed to make inference in option pricing and risk management. Tsay (2009)

In this chapter, we demonstrate the efficiency of the Monte-Carlo significance test by considering a real data set. we consider the case of running the analysis on the daily log closing returns of Ooredoo Qatar. The data has been retrieved from the website link https://www.ooredoo.com/en/investtors/share_information/historical-share-prices/ over the market days starting from 11/09/2008 till 26/02/2020 (excluding weekends). The series length has 2984 days. For the aim of the analysis, we analyze the returns of Ooredoo instead of the prices. So, the time series data will be more stationary. Also, log return is used instead of returns, as using log eliminates the non-stationary properties of the data set in a way of making the data more stable. The plot in Figure 1 shows the daily log returns versus time. As shown in the figure the mean of log return is constant and almost zero, but the volatility gets higher between 2012-2014.

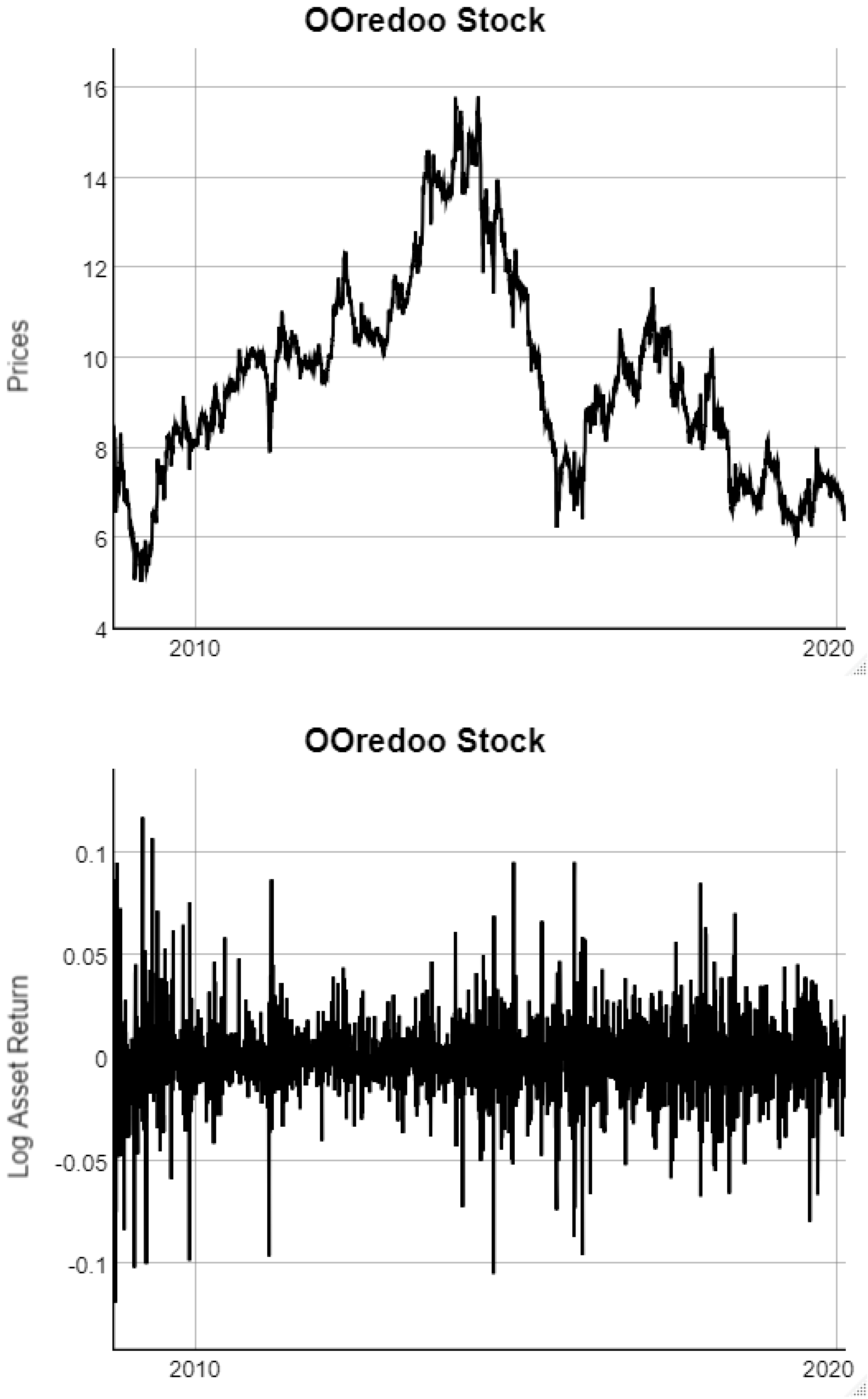

Figure 1: Ooredoo daily Stock price and log returns from 2008-2020

Figure 1 shows the daily stock price of Ooredoo from 2008-2020 and the second figure shows the log return from 2008-2020. Moreover, Figure 2 and Figure 3 shows the log-returns using a histogram and Q-Q Plot. Both figures confirm that the data does not follow a normal distribution. That can be also checked from the value of the Kurtosis. Where kurtosis checks and measures the spread of the distribution if it's too peaked that's mean that the distribution is narrow and most of the responses are in the middle. Three types of kurtosis can be shown by a set of data: Mesokurtic, Leptokurtic, and Platykurtic. Starting with Mesokurtic distribution, which is the nearest to the normal distribution which means that the maximum value of the distribution is similar to normal distribution. The second distribution is the Leptokurtic, where it shows more values in the tail of the distribution and mostly close to the mean value. An example of Leptokurtic is the T-distribution with a small degree of freedom. The final type of distribution is the Platykurtic, where it has the shortest tail and shows fewer values in the tails with a fewer value near to mean. An example of this is the uniform distribution. The descriptive statistics table for Ooredoo stock is presented in the below table:

Table 8. Descriptive statistics for Ooredoo stock price and log-return

Variable	mean	median	Mode	Std	IQR	Skewness	Kurtosis
Price	9.475	9.330	9.018	2.224	2.914	0.557	2.835
Log return	0.000	0.000	-0.004	0.017	0.012	-0.047	11.422

From the descriptive statistics, the value of Kurtosis is 11.422 which is larger than normal distribution kurtosis usually equal to 3. The result indicates that the distribution of the log-returns of Ooredoo follows a Leptokurtic distribution and not a normal distribution. The skewness value is -0.047 which is close to zero. The distribution of prices is right skewed with skewness equal 0.55 is different from the skewness of normal distribution which is equal zero. Shapiro-Wilk test of normality rejects the null hypothesis of having normal distribution for both price and log-return series (W = 0.873, P-value < 2.2e-16). Kolmogorov-Smirnov normality test also rejects the null hypothesis of having normal distribution (D=0.471, P-value < 2.2e-16). Aderson-Darling test (updated version of Kolmogorov-Smirnov test which gives more weight to the tails of the distribution) also rejects the null hypothesis of having normal distribution (A = 2974, p-value < 2.2e-16). The deviation from normality also could be seen in the qqplot which is presented in Figure 3.

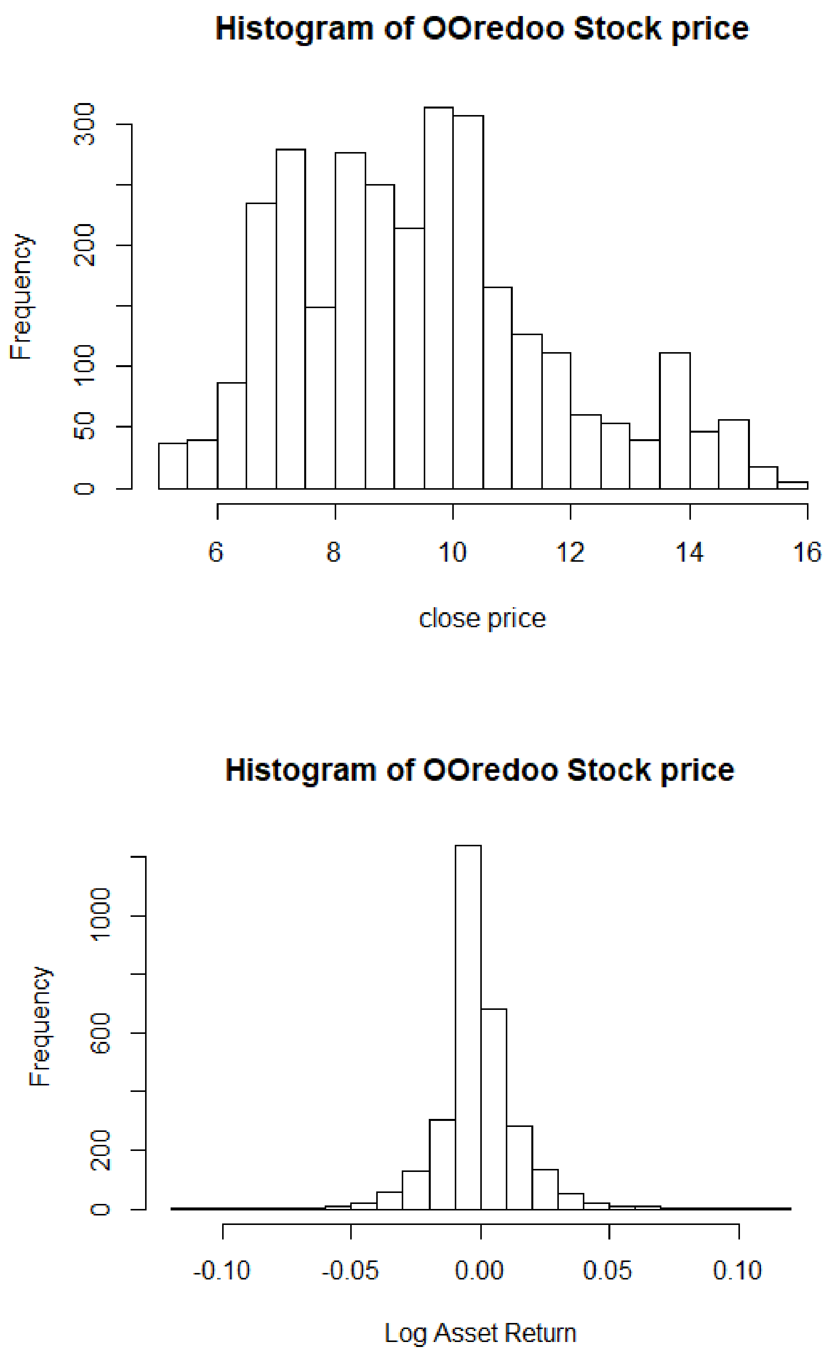

Figure 2: Histogram of Stock price and log-returns of Ooredoo

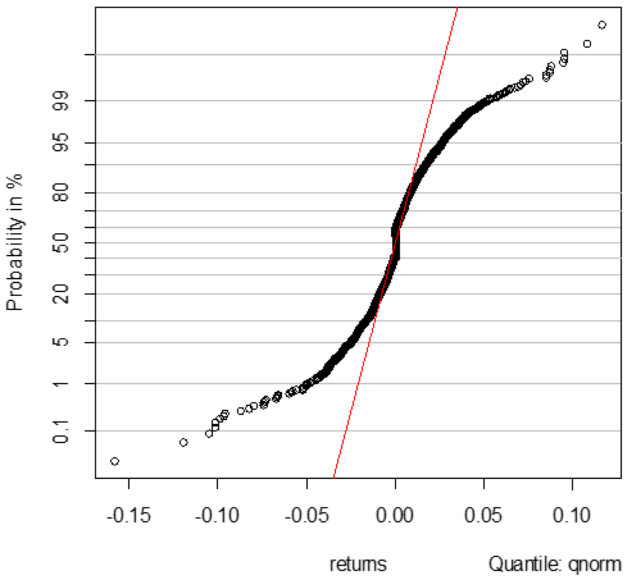

Figure 3: Q-Q plot of Ooredoo log returns

To fit the model to Ooredoo data, we checked whether the data model can be estimated by ARIMA, ARCH, and GARCH. We used the information criteria (AIC) proposed by Akaike (1974) to select the best model. The model with the smaller information criteria value is assumed to be the best model. We start first by fitting the ARIMA model under the null hypothesis that there is no trend. After running the R-code the results show that no ARIMA has been detected. On the other side, I fit ARCH and GARCH, and I tried the following models: GARCH *(1,0)*, GARCH *(1,1)*, GARCH *(2,0)*, GARCH *(2,1)*, GARCH *(2,2)* and GARCH *(3,0)*.

The results in Table 10 compare the AIC values for the six models. The results indicate that the best model is GARCH *(1, 1)*, as it has the lowest AIC. Where Akaike equals to -5.6668 and BIC equal to -5.6587, so I choose GARCH *(1,1)* to continue my analysis.

Table 9. Information criteria for the 6 GARCH models

	GARCH (1,0)	GARCH (1,1)	GARCH (2,0)	GARCH (2,1)	GARCH (2,2)	GARCH (3,0)
Akaike	-5.5757	-5.6668	-5.6114	-5.6666	-5.6665	-5.6287
Bayes	-5.5696	-5.6587	-5.6033	-5.6565	-5.6544	-5.6186
Shibata	-5.5757	-5.6668	-5.6114	-5.6666	-5.6665	-5.6287
Hannan-Quinn	-5.5735	-5.6639	-5.6085	-5.6629	-5.6622	-5.6250

To check the adequacy of the chosen model GARCH *(1, 1)*, we apply the Monte-Carlo tests of four tests and compare the results. The four tests are Li-Mak (1994) test, weighted Li-Mak(1994) which is proposed by Fisher and Colin (2012), Cross-correlation test proposed by Psaradakis and Vávra (2019) and lastly the Monte-Carlo version of Mahdi (2020) test. Table *10* shows the p-value of the Monte Carlo process for *L(b,m)*, $L_w(b,m)$, Q_{rs} and C_m. The results are recorded at different lags from lag 5-50. As the p-value gets closer to 0.05 or less that shows that the model is not good. We reject the null hypothesis and we should look for a complicated model, not the GARCH model. On the other hand, if a p-value greater than 0.05, then this suggests that the model is good. Table 10 show the results of the four tests where two of the test accept the model and believe that it is a good model. But the other two test reject the model and believe the model is not good and something is hidden as their p-value is smaller than 0.05 in most or all the lags.

Table 10. The p-value for the tests L(b,m), $L_W(b,m)$, Q_{rs} ,C_m.

Lag	L(b,m)	L_W(b,m)	Q_{rs}	C_m
5	0.064453	0.6037	0.6745	$1.4049e^{-12}$
10	0.064724	0.6965	0.8286	$4.5104e^{-15}$
15	0.050050	0.3625	0.7697	$1.4927e^{-16}$
20	0.014009	0.2547	0.6656	$1.0259e^{-19}$
25	0.028626	0.2633	0.4645	$3.3564e^{-22}$
30	0.030019	0.2499	0.4167	$1.5279e^{-23}$
35	0.030981	0.1652	0.3390	$5.5945e^{-25}$
40	$2.3072e^{-05}$	0.0659	0.1151	$2.0973e^{-26}$
45	$6.1240e^{-05}$	0.0221	0.1805	$2.0973e^{-27}$
50	$1.0277e^{-04}$	0.0098	0.2437	$1.9356e^{-27}$

4.2 Application part 2

In the second part, we used the same data for Ooredoo but the analysis is divided into two parts: before the blockade of Qatar from the airspace of Saudi Arabia, UAE, Bahrain and Egypt and after. The blockade starts on 5/June/2017 and still going till now. The dataset is divided into two parts before and after which will allow me to see the effect the blockade had on

Ooredoo returns. The descriptive statistics table for Ooredoo stock before and after blockade is presented in the below table:

Table 11. Descriptive statistics for Ooredoo stock before and after blockade

Variable	Blockade	mean	median	Mode	Standard deviation	IQR	Skewness	Kurtosis
Price	Before	10.060	9.960	10.058	2.185	2.650	0.282	2.861
	After	7.610	7.300	7.241	0.964	1.471	0.679	2.412
Log return	Before	0.000	0.000	-0.004	0.017	0.011	-0.065	13.121
	After	-0.001	0.000	-0.004	0.017	0.014	0.010	6.253

The descriptive statistics for Ooredoo log-returns before blockade has kurtosis equal 13.12 and after blockade it is equal 6.25 both are deviated from the kurtosis of normal distribution and are Leptokurtic. The skewness of the log-return are -0.065 and 0.01 for before and after blockade which are close to zero. Stock prices are skewed to right for both before and after blockade. But the skewness increases clearly after blockade event. (0.282, 0.679). Stock prices and log return for both before and after blockade do not follow a normal distribution. Shapiro-Wilk test of normality rejects the null hypothesis of having normal distribution for both split of before and after blockade. The results of the test for log-return before and after blockade are W=0.847, P-value < 2.2e-16 and W=0.942, P-value = 5.235e-16 respectively. Anderson-Darling test of normality also rejects the null hypothesis of having normal distribution for both splits of the time series for before and after blockade, the results of this test for both splits are A = 2244.7, p-value < 2.2e-16 and A = 702.09, p-value < 2.2e-16 respectively.

Figure 4 shows the log returns for Ooredoo before and after the blockade, it showed clearly from the plot that Ooredoo has been affected by the crisis and returns have decreased.

Figure 4 shows the log-returns plot against time. It's clear from the plot that Ooredoo has lower returns after the blockade than before.

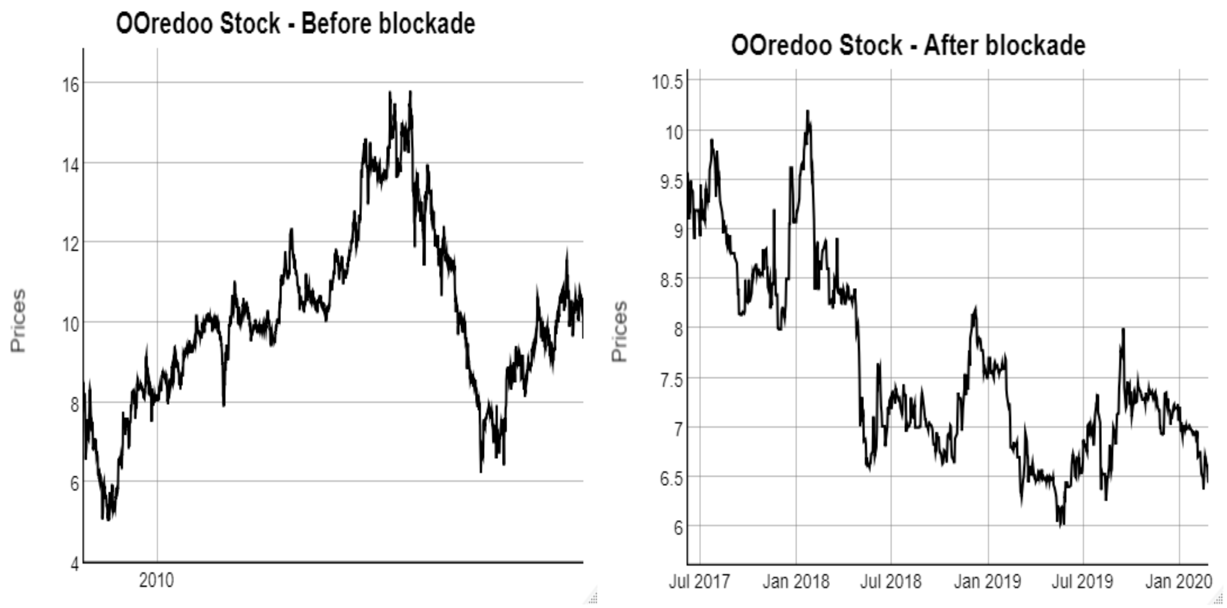

Figure 4. The Stock price before and after blockade

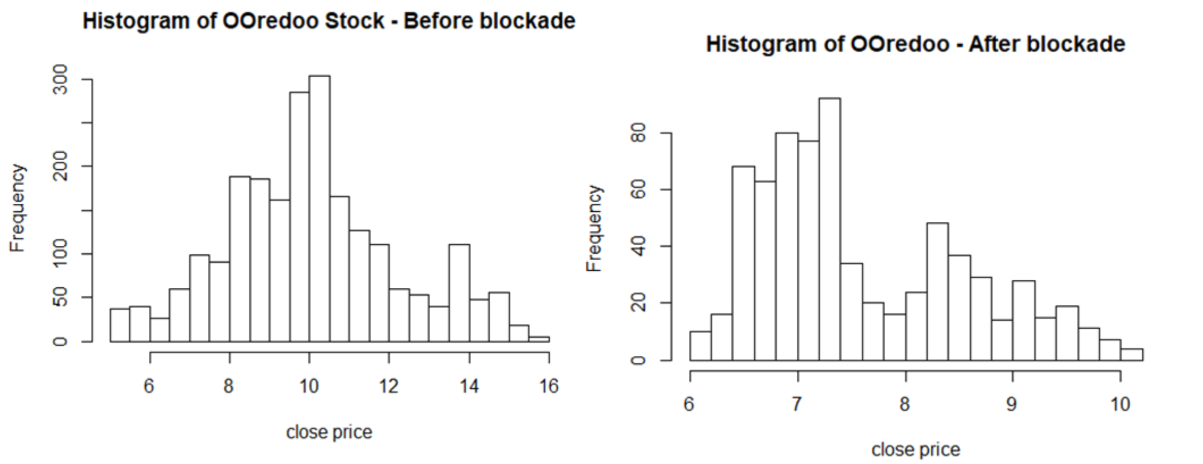

Figure 5. Histogram of Stock price before and after blockade

The same analysis has done in the first part will be repeated the only difference is that the data is divide into two parts.

Figure 6 and Figure 7 reflect the results for the log-returns before and after the blockade. As shown in the figures before blockade the returns had a lower variability than after. This indicates that Ooredoo returns have been affected strongly by the crisis. The figure shows in both cases before and after still the retunes does not follow a normal distribution.

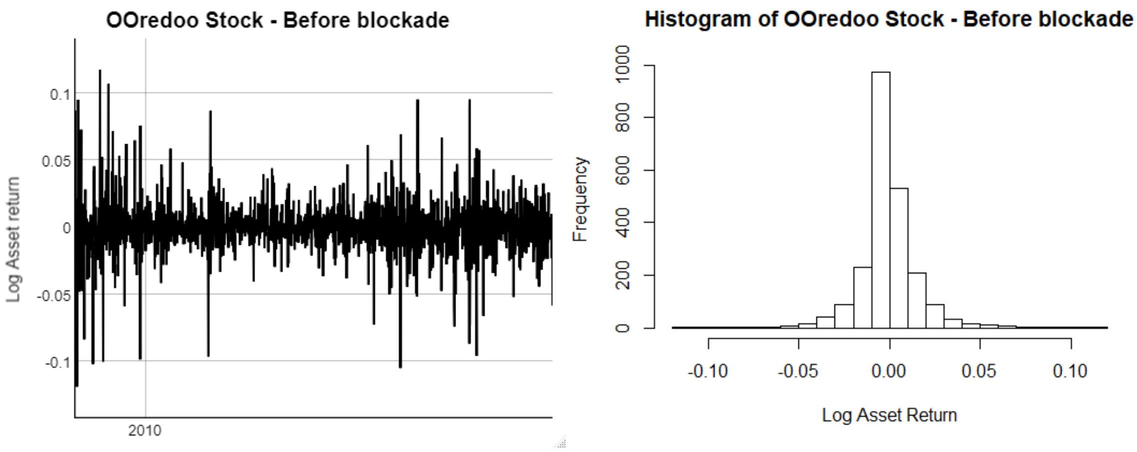

Figure 6. Daily log returns plot and histogram for Ooredoo returns before the blockade

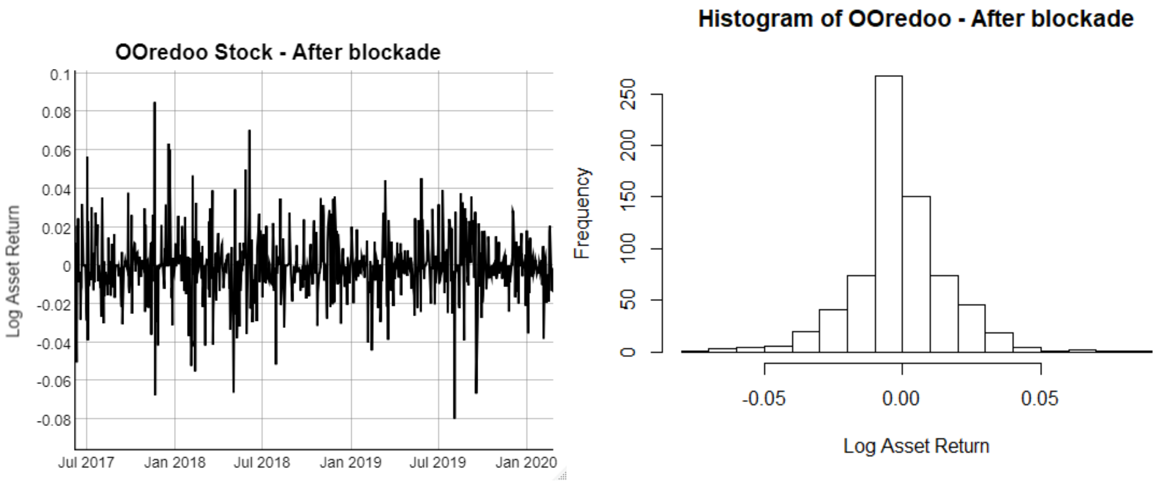

Figure 7. Daily log returns plot and histogram for Ooredoo returns after blockade

We start first by fitting the ARIMA model under the null hypothesis that there is no trend. After running the R-code the results show that no ARIMA has been detected. On the other side, we fit ARCH and GARCH, and we tried the following models: GARCH *(1,1)*, GARCH *(2,0)*, GARCH *(2,1)*, GARCH *(2,2)* AND GARCH *(3,0)*, GARCH *(3,1)*.

The results in Table 12 compare the information criteria values for the six models. The results indicate that the best model is GARCH *(1, 1)*, as it has the lowest AIC. Where Akaike equals to 1.6874 and BIC equal to 1.7130, so we choose GARCH *(1,1)* to continue my analysis.

Table 12. Information criteria vales for GARCH models

	GARCH (1,1)	GARCH (2,0)	GARCH (2,1)	GARCH (2,2)	GARCH (3,0)	GARCH (3,1)
Akaike	1.6874	1.6908	1.6918	1.6946	1.6944	1.6977
Bayes	1.7130	1.7165	1.7238	1.7331	1.7264	1.7361
Shibata	1.6873	1.6908	1.6917	1.6945	1.6943	1.6975
Hannan-Quinn	1.6973	1.7007	1.7042	1.7095	1.7067	1.7125

To check the adequacy of the chosen model GARCH *(1, 1),* we apply the Monte-Carlo tests of four tests and compare the results. The four tests are Li-Mak (1994) test, weighted Li-Mak(1994) which is proposed by Fisher and Colin (2012), Cross-correlation test proposed by Psaradakis and Vávra (2019) and lastly the Monte-Carlo version of Mahdi (2020) test. Table 13 shows the results of the four test the first three tests in most of the lags they accept the model and think it's a good model. But for the Cm test is the only test that reject the model as it believe that something is hidden and the model is not working perfectly. Which shows that the Cm test has found something that other test did not and the results shows consistency as it reject the model at all the lags. But for the first test (L(b,m)) it suffer from consistency as it reject the model in some lags and accept the model in others lags which is not a good sign of testing. In general, in statistics we believe more with the model that reject more as it is easy to accept but it's very hard to reject and figure out something that others didn't.

Table 13. P-value for $L(b,m)$, $L_W(b,m)$, Q_{rs} ,C_m.

Lag	L(b,m)	$L_W(b,m)$	Q_{rs}	C_m
5	0.421	0.445	0.462	0
10	0.799	0.746	0.739	0
15	0.408	0.603	0.633	0
20	0.610	0.616	0.677	0
25	0.0225	0.430	0.552	0
30	0.067	0.288	0.439	0
35	0.111	0.241	0.418	0
40	0.158	0.217	0.411	0
45	0.271	0.223	0.430	0
50	0.277	0.227	0.457	0

CHAPTER 5: CONCLUSION

5.1 Conclusion

Various approaches for testing the adequacy of the model for both linear and nonlinear models were compared in this study with the one which released lately by Mahdi (2020). The portmanteau test on simulated data by AR*(p)*, MA*(p)*, ARMA*(p,q)*, ARCH and GARCH process were compared together using Monte Carlo simulation. The results for power of the tests shown clearly that the lately extended portmanteau test by Mahdi (2020) for testing the adequacy of the fit had the best performance comparing with other tests in 15 out of 24 models including both linear and nonlinear models. In 5 of the models, the statistics presented by Peña and Rodríguez (2002, 2006) for the squared residuals had the best performance. In 4 models the test statistics of Fisher and Gallagher (2012) which is a weighted statistics of McLeod and Li had the highest power compared with other tests. The presented statistics of Mahdi (2020) also were tested on the real-time series data (Ooredoo Qatar data). Four lately extended tests were implemented to check the serial correlation in the residuals of the fitted model (for $L(b,m)$, $L_W(b,m)$, Q_{rs} , C_m). ARMA and GARCH models were fitted to the Ooredoo data and it was seen that the GARCH(1,1) model had the lowest AIC. The best model found by AIC was tested by these four portmanteau tests and two tests of Li-Mak and Mahdi test shown that the model is not adequate since there is still a significant serial correlation in the residuals. In the final step, the Ooredoo data was split into two parts before the crisis of blockade in 2017 and after the blockade. The data was analyzed and the best model was seen to be GARCH*(1,1)* according to the AIC. The four mentioned tests were implemented on the resultant model. The only test which still rejects the null hypothesis and shows that the model is not adequate is Mahdi test and the other three tests agree that there is no serial correlation in the residuals and since the residuals of the GARCH*(1,1)* model do not follow a normal distribution, it means that the Mahdi (2020) test could correctly detect the adequacy of the fit compared with the other tests.

5.2 Suggestions for Further Study

Some suggestions that could be helpful and useful to implement in future work is to extend the Cm test for generalized correlations (autocorrelation between residuals at different powers). In addition, the Cm test statistic may be modified to check the adequacy of the fitted model in multivariate data and with other types of data, for example, environmental. As in this thesis we focused on time series data. Moreover, this test could be to implement it on seasonal time series data.

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