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# Dimensional Regularization of the Spatial wave function for a singular contact interaction in the Relativistic Schrodinger Equation

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**Abstract.** Based on our previous work in PRD 89, 125023 (2014), we stress here (for the first time) the regularization of the spatial wave function for the  $\delta$ -contact interaction within the relativistic Schrodinger equation. The D-dimensional inverse Fourier transform has been utilized to map the momentum-space wave function to the spatial one. To regularize the logarithmic blow up of the wave function as  $x \rightarrow 0$ , we employed the dimensional regularization technique. We assert that although the technique has been used here for the bound state only, the form of the scattering states in momentum space assures the reliability of the same technique to regularize the spatial scattering wave functions.

In particle physics, the asymptotic freedom property plays a vital role in describing strong interactions. This property is responsible for the confinement of quarks inside protons (for instance). In fact, the need for a theory with such property was based on the experimental results from SLAC [1] where it has been discovered that fast electrons scatter from protons as if they have scattered from quasi-free, point-like constituents inside the protons. At that time there were no known theory that can describe the scattering pattern as QED shows an opposite behavior. This led to the formulation of quantum chromodynamics, a non-abelian gauge field theory with the group structure  $SU(3)$ . In QCD, the number of colors is equal to or greater than three to secure the existence of the asymptotic freedom [6, 8, 7]. In 1975, Symanzik [2] advocated the simpler one-component ( $-\phi^4$ ) scalar field theory and later the asymptotic freedom of this theory has been stressed [3, 4, 5]. Other studies also stressed the existence of quantum field-like properties in non-relativistic quantum mechanical theories [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Very recently, we discovered the existence of such property in a relativistic Schrodinger equation with contact interactions [19, 20].

In the relativistic Schrodinger equation where the kinetic term takes the form  $\sqrt{p^2 + m^2}$ , certain precautions should be taken into account. According to Leutwyler's non-interaction theorem [21], the potential term to be added to that kinetic term should not let the theory violating the principles of the special theory of relativity. In fact, the no-go theorem of Leutwyler admits the singular contact interactions and thus there is no logic reason of excluding them from relativistic quantum mechanical studies. The study of the singular contact interactions within the context of relativistic Schrodinger equation enables us to avoid certain problems that do exist in either Klein-Gordon or Dirac equations. For instance, the existence of Ghost states in Klein-Gordon equation and the existence of holes in Dirac equation represent main problems



in these theories when considering them as relativistic quantum mechanical theories. These theories are in fact successful when considered as quantum field ones.

In Refs.[19, 20] we studied two types of one dimensional contact interactions within the relativistic Schrodinger equation. Although the two potential terms are different, a type of universality has been shown up. The universality characteristic is manifested in finding that the two theories shares the same properties. However, the coupling in the two theories are of different mass dimensions which reflects the need for using two different regularization techniques. For the  $\delta$ - potential, the coupling is dimensionless and thus the theory contains only logarithmic type of divergences. On the other hand, the coupling in the  $\delta'$ - potential has a length dimension and the theory contains logarithmic, linear and quadratic type of divergences. For logarithmic divergences both dimensional and cut off regularization lead to the same results but for other type of divergences, dimensional regularization may not reflect the correct scale dependance ( for instance for the scalar  $\phi^4$  field theory in  $2 + 1$  space-time dimensions, linear divergences will not appear in using dimensional regularization). In our work in Refs. [19, 20], the two theories have been regularized in momentum space and the spatial wave function has shown up a logarithmic blow up as  $x$  approaches zero. In this work, we stress this point and show that using dimensional regularization, one can show that the blow up of the wave function as  $x \rightarrow 0$  can be regularized using dimensional regularization. This point has not been stressed before and represents the main point of our work here.

To start, consider the relativistic Schrodinger equation in  $D$  dimensions of the form:

$$\sqrt{p^2 + m^2}\Psi(x_1, x_2, \dots, x_D) + \lambda\delta(x_1, x_2, \dots, x_D)\Psi(x_1, x_2, \dots, x_D) = E\Psi(x_1, x_2, \dots, x_D). \quad (1)$$

We have in momentum space the following quantities:

$$\Psi(x_1, x_2, \dots, x_D) = \frac{1}{(2\pi)^D} \int d^D p \tilde{\Psi}(p) \exp(i\vec{p} \cdot \vec{x}), \quad (2)$$

$$\Psi(0, 0, \dots, 0) = \frac{1}{(2\pi)^D} \int d^D p \tilde{\Psi}(p), \quad (3)$$

$$\tilde{\Psi}(p) = \frac{\lambda\Psi(0, 0, \dots, 0)}{(E_B - \sqrt{p^2 + m^2})}. \quad (4)$$

Here  $E_B$  represents the bound state energy. Since  $\Psi(0, 0, \dots, 0)$  is a quantity in position space, it would be more plausible to express  $\tilde{\Psi}_B(p)$  in terms of the momentum representations of all quantities. So we put:

$$\tilde{\Psi}(p) = \frac{\lambda}{2\pi} \frac{\int d^D p' \tilde{\Psi}(p')}{(E_B - \sqrt{p^2 + m^2})}. \quad (5)$$

In fact, this is a simple integral equation for which  $\tilde{\Psi}(p)$  given by:

$$\tilde{\Psi}(p) = \frac{A\lambda}{2\pi (E_B - \sqrt{p^2 + m^2})} \quad (6)$$

represents a solution, where  $A$  is a normalization constant. This solution will satisfy the integral equation since when substituted back into integral equation we get:

$$\tilde{\Psi}(p) = \lambda A \frac{\int d^D p' \frac{\lambda}{2\pi (E_B - \sqrt{(p')^2 + m^2})}}{2\pi (E_B - \sqrt{p^2 + m^2})}, \quad (7)$$

we can employ the gap equation in Ref.[19] that reads:

$$\int \frac{dp'}{2\pi \left( E_B - \sqrt{(p')^2 + m^2} \right)} = \frac{1}{\lambda}. \quad (8)$$

Thus the form  $\tilde{\Psi}(p)$  in Eq.(6) represents a solution of the integral equation. This form of  $\tilde{\Psi}(p)$  suggests that the regularization of  $\lambda$  will certainly regularize the wave function in momentum space. Using dimensional regularization we get:

$$\begin{aligned} \frac{m^{D-1}}{\lambda(\epsilon)} &= \frac{1}{(2\pi)^D} \int d^D p \frac{1}{E_B - \sqrt{p^2 + m^2}} = I(E_B) \\ &= -\frac{1}{(2\pi)^D} \int \frac{d^D p}{\sqrt{p^2 + m^2}} - \frac{1}{(2\pi)^D} \int \sum_{n=1}^{\infty} d^D p \frac{(E_B)^n}{(p^2 + m^2)^{\frac{n+1}{2}}} \\ &= -\frac{1}{(2\pi)^D} \int \frac{d^D p}{\sqrt{p^2 + m^2}} - \frac{E_B}{2\pi \sqrt{m^2 - E_B^2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right) \\ &= -\frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(\frac{1-D}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left( \frac{1}{m^2} \right)^{\frac{1-D}{2}} - \frac{E_B}{2\pi \sqrt{m^2 - E_B^2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right) \\ &\propto \frac{1}{\epsilon} + (\text{finite terms}), \quad \epsilon = D - 1. \end{aligned} \quad (9)$$

The UV divergences in momentum space will have its manifestation in position-space as small- $x$  divergences. The position-space wave function is obtained via Fourier inverse transformation of  $\tilde{\Psi}(p)$ . Since the inverse Fourier transform is an integration over momentum, one can aim to employ dimensional regularization to get a regularized wave function; a wave function that is finite everywhere but scale dependent on the other hand. To do this let us consider the following:

$$\begin{aligned} \tilde{\Psi}(p) &= \frac{A\lambda}{2\pi \left( E_B - \sqrt{p^2 + m^2} \right)} \\ &= -\frac{1}{(2\pi)^D} \frac{A\lambda}{\sqrt{p^2 + m^2}} - \frac{A\lambda}{(2\pi)^D} \sum_{n=1}^{\infty} \frac{(E_B)^n}{(p^2 + m^2)^{\frac{n+1}{2}}}. \end{aligned} \quad (10)$$

In 1-dimension ( $D = 1$ ), the spatial wave function takes the form:

$$\begin{aligned} \Psi(x) &= \frac{1}{(2\pi)^D} \int d^D p \frac{\exp(-i \vec{x} \cdot \vec{p})}{E - \sqrt{p^2 + m^2}} \\ &= -\frac{1}{(2\pi)^D} \int d^D p \frac{\exp(-i \vec{x} \cdot \vec{p})}{(p^2 + m^2)^{\frac{1}{2}}} \\ &\quad - \frac{A\lambda}{\sqrt{\pi}} \sum_{n=1}^{\infty} \left( \frac{E_B}{m} \right)^n \left( \frac{m|x|}{2} \right)^{n/2} \frac{K_{n/2}(m|x|)}{\Gamma\left(\frac{n+1}{2}\right)}, \end{aligned} \quad (11)$$

where  $K_{n/2}$  are the modified Bessel functions of second kind. In fact, every term in the summation above is finite in the limit  $D \rightarrow 1$  but the integral of the form:

$$T_0(x) = -\frac{1}{(2\pi)^D} \int d^D p \frac{\exp(-i \vec{x} \cdot \vec{p})}{(p^2 + m^2)^{\frac{1}{2}}} \quad (12)$$

is logarithmic divergent by power counting and thus needs to be regularized. To do this let us consider:

$$\vec{x} \cdot \vec{p} = |x| |p| \cos(\theta). \tag{13}$$

In  $D$  dimensions, the volume element is

$$d^D p = p^{D-1} \sin^{D-2}(\phi_1) \sin^{D-3}(\phi_2) \cdots \sin(\phi_{D-2}) dr d\phi_1 d\phi_2 \cdots d\phi_{D-1}$$

where  $\theta \equiv \phi_1$ . (14)

Then,

$$\begin{aligned} & \frac{1}{(2\pi)^D} \int d^D p \frac{\exp(-i \vec{x} \cdot \vec{p})}{(p^2 + m^2)^{\frac{1}{2}}} \\ = & \frac{1}{(2\pi)^D} \int d^D p \frac{\exp(-i |x| |p| \cos(\phi_{D-2}))}{(p^2 + m^2)^{\frac{1}{2}}} \\ = & \frac{1}{(2\pi)^D} \int_{\phi_{D-1}=0}^{2\pi} \int_{\phi_{D-2}=0}^{\pi} \cdots \int_{\phi_1=0}^{\pi} \int_0^{\infty} p^{D-1} \sin^{D-2}(\phi_1) \sin^{D-3}(\phi_2) \cdots \sin(\phi_{D-2}) \times \\ & \times \frac{\exp(-i |x| |p| \cos(\phi_{D-2}))}{(p^2 + m^2)^{\frac{1}{2}}} dp d\phi_1 d\phi_2 \cdots d\phi_{D-2} d\phi_{D-1} \end{aligned} \tag{15}$$

Now, the integral  $I_n$  over the angles gives:

$$\begin{aligned} I_n &= \int_{\phi=0}^{\pi} \sin^n(\phi) d\phi = \sqrt{\pi} \frac{\Gamma(\frac{1+n}{2})}{\Gamma(\frac{n}{2} + 1)}, \\ I_n I_{n-1} &= \frac{2\pi}{n}, \\ I_{n-2} I_{n-3} &= \frac{2\pi}{n-2}. \end{aligned} \tag{16}$$

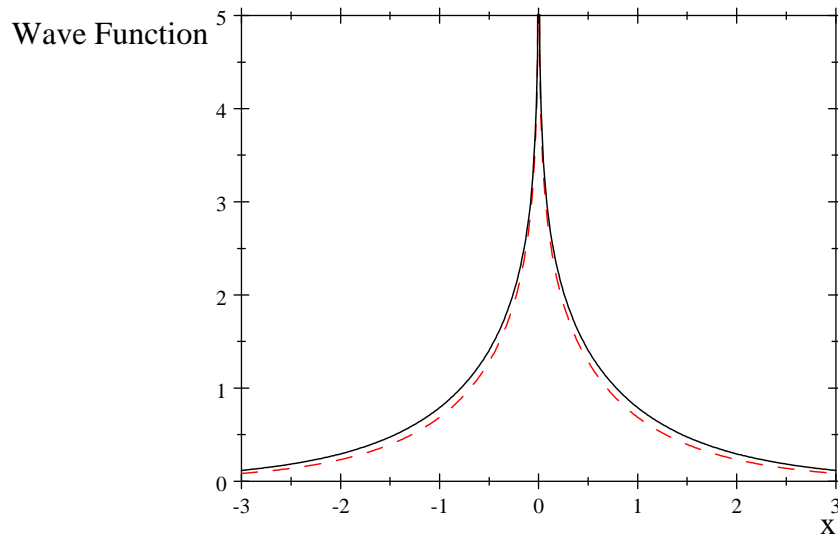
Or in general, the inverse Fourier transform of the wave function in  $D$ -dimensions results in the following form:

$$\Psi(x) = Am^{1-D} \lambda (2\pi)^{\frac{D}{2}} \sum_{n=1}^{\infty} E_B^{n-1} \frac{2^{-n/2} n |x|^{-D/2} \left(\frac{1}{m^2}\right)^{\frac{1}{4}(-D-n)} \left(\frac{m^2}{|x|}\right)^{-n/2} K_{\frac{D-n}{2}} \left(\frac{|x|}{\sqrt{\frac{1}{m^2}}}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}. \tag{17}$$

This sum is rapidly convergent as it can be realized from Fig.1. In fact, this form represents the solution of the original relativistic Schrodinger equation in  $D$ -dimensions which has not been obtained before. In  $D = 1$ , the wave function is logarithmically divergent at the origin while for the  $D = 2$  case the wave function possesses logarithmic as well as linear type of divergences (in momentum space but  $1/x$  in position space). In  $D = 3$  an extra quadratic type of divergences appears. The divergences in higher dimensions (higher than 1) need more studies which will be done in another work.

For  $|x| = 0$ , one can re-sum the series to get:

$$\Psi(0) = Am^{1-D} \lambda \frac{\pi^{\frac{D-1}{2}} \left(\frac{1}{m^2}\right)^{-D/2} \left( \begin{aligned} & -E_B^2 \Gamma\left(\frac{1-D}{2}\right) {}_2F_1\left(1, \frac{1}{2} - \frac{D}{2}; \frac{1}{2}; \frac{E_B^2}{m^2}\right) \\ & + m^2 \Gamma\left(\frac{1-D}{2}\right) {}_2F_1\left(1, \frac{1}{2} - \frac{D}{2}; \frac{1}{2}; \frac{E_B^2}{m^2}\right) \\ & + \sqrt{\pi} E_B m \Gamma\left(1 - \frac{D}{2}\right) \left(1 - \frac{E_B^2}{m^2}\right)^{D/2} \end{aligned} \right)}{\sqrt{m^2} (m^2 - E_B^2)}. \tag{18}$$



**Figure 1.** The bound state wave function for the summation of the first three terms (red) compared to the first seven terms (black) for  $E_B = 0.5$  and  $m = 1$ . The figure shows a rapid convergence of the series.

Expanding in powers of  $\epsilon = D - 1$ , we obtain:

$$\psi(0) \propto A\lambda \left( \frac{1}{\epsilon} + \text{finite terms} \right). \quad (19)$$

Since in Eq.(9)  $\frac{1}{\lambda} \propto \left( \frac{1}{\epsilon} + \text{finite} \right)$ , then as  $\epsilon \rightarrow 0$ ,  $\psi(0) \propto A$ . This means that the dimensional regularization method regularizes the wave function in position space and no need for extra renormalization conditions.

In fact, the logarithmic behavior of the wave function for small  $|x|$  values for the unregularized wave function in our paper in Ref. [19] has been reflected here as  $\ln \frac{1}{m}$  in the regularized wave function (position has an  $m^{-1}$  dimension in natural units). This result is very important as one is not in a need for other renormalization conditions and the coupling renormalization condition used in our paper in Ref. [19] is the only needed one to turn all physical quantities finite. Although in the above discussion we stressed the regularization of the position-space bound state wave function, the same argument can be extended for the scattering states wave functions. In fact, the scattering state wave function in momentum space is given by [19]:

$$\begin{aligned} \tilde{\Psi}_E(p) &= \delta(p - \sqrt{E^2 - m^2}) + \delta(p + \sqrt{E^2 - m^2}) + \tilde{\Phi}_E(p), \\ \tilde{\Phi}_E(p) &= \frac{\lambda}{\pi} \frac{1 + \pi\Phi_E(0)}{E - \sqrt{p^2 + m^2}}. \end{aligned} \quad (20)$$

While the first two terms in  $\tilde{\Psi}_E(p)$  result in two plane waves in position space, the third term has a similar behavior as the bound state. Accordingly, the inverse Fourier transform of the  $\tilde{\Phi}_E(p)$  is logarithmic divergent and can thus be regularized in a similar way.

To conclude, we considered the singular contact interaction within the frame work of the relativistic Schrodinger equation. The theory shows up rigorous quantum-field-like properties such as asymptotic freedom and dimensional transmutation. Since the theory bears divergences in some physical quantities, dimensional regularization has been employed to regularize the

theory. In regularizing the theory in momentum space, the coupling renormalization condition has been utilized. To obtain the spatial wave functions, a direct inverse Fourier transform can be applied but then the position-space wave functions blow up as  $|x| \rightarrow 0$ . This might lead to the conclusion that the renormalization condition used for the coupling in momentum space is not sufficient to turn all quantities finite in position space. However, based on the fact that position space is totally equivalent to the momentum space, one should not obtain divergent quantities in position space from already regularized quantities in momentum space. To lift this conflict, we employed the  $D$ -dimensional inverse Fourier transform to map quantities from momentum space to position space. In doing this, we found that the regularized wave functions in position space are finite but rather scale dependent.

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