



# Condenser capacity and hyperbolic diameter

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## ABSTRACT

Given a compact connected set  $E$  in the unit disk  $\mathbb{B}^2$ , we give a new upper bound for the conformal capacity of the condenser  $(\mathbb{B}^2, E)$  in terms of the hyperbolic diameter  $t$  of  $E$ . Moreover, for  $t > 0$ , we construct a set of hyperbolic diameter  $t$  and apply novel numerical methods to show that it has larger capacity than a hyperbolic disk with the same diameter. The set we construct is called a Reuleaux triangle in hyperbolic geometry and it has constant hyperbolic width equal to  $t$ .

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## 1. Introduction

One of the famous problems of geometry is the problem of maximizing the volume of a geometric body given its surface area. This *isoperimetric problem* is a constrained extremal problem connecting two domain functionals, the volume and the surface area of the domain in question. Other than this specific question, there are several kinds of constrained extremal problems motivated by geometry and mathematical physics that can be referred to as isometric problems.

Already seventy years ago, G. Pólya and G. Szegő studied isoperimetric problems in their famous book [20], which inspired numerous later authors. They specifically devoted a lot of attention to isoperimetric problems involving condenser capacity. Condenser capacities are important tools in the study of partial differential equations, Sobolev spaces, integral inequalities, potential theory, see V. Maz'ya [15] and J. Heinonen, T. Kilpeläinen, and O. Martio [10]. Since the extremal situations for the isoperimetric problems often reflect symmetry, various symmetrization procedures can be used as a method for analyzing

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isoperimetric problems, see A. Baernstein [1]. Furthermore, in his pioneering work [5], V.N. Dubinin systematically developed capacity related methods and gave numerous applications of condenser capacities and symmetrization methods to classical function theory. Capacity is also one of the key techniques in the theory of quasiconformal and quasiregular maps in the plane and space [7,9,21,22].

An open, connected and non-empty set  $G$  is called a *domain* and if  $E \subset G$  is a compact non-empty set, then the pair  $(G, E)$  is a *condenser*. The *capacity* of this condenser is defined by

$$\text{cap}(G, E) = \inf_u \int_G |\nabla u|^n dm, \quad (1.1)$$

where the infimum is taken over the set of all  $C_0^\infty(G)$  functions  $u : G \rightarrow [0, \infty)$  with  $u(x) \geq 1$  for all  $x \in E$  and  $dm$  is the  $n$ -dimensional Lebesgue measure. Below, we often choose  $n = 2$  and focus on the special case where  $G \subsetneq \mathbb{R}^2$  is simply connected and  $E$  is a continuum.

By classical results the capacity decreases under a geometric transformation called symmetrization [1, Ch. 6, p. 215], [5], [7, Thm 5.3.11],

$$\text{cap}(G, E) \geq \text{cap}(G_s, E_s), \quad (1.2)$$

where  $(G_s, E_s)$  is the condenser obtained by one of the well-known symmetrization procedures, such as the spherical symmetrization or the Steiner symmetrization. While finding the explicit formula for the capacity (1.1) is usually impossible, the lower bound (1.2) can be often estimated or given explicitly [5], [7, pp. 180-181], [9, Chapter 9].

Our aim in this paper is to find upper bounds for the condenser capacity, when  $n = 2$ ,  $G$  is a simply connected domain, and  $E$  is a connected compact set. Numerous bounds are given in the literature in terms of domain functionals, such as the area of  $G$ , the diameter of  $E$  and the distance from  $E$  to the boundary  $\partial G$  [5,7,9,15,17,21,22]. While these kinds of bounds have numerous applications as shown in the cited sources, these bounds do not reflect the conformal invariance of  $\text{cap}(G, E)$ . We apply the conformally invariant hyperbolic metric in this paper and therefore our main results are conformally invariant.

By the conformal invariance of the capacity and the hyperbolic metric, we may assume without loss of generality that the domain  $G$  is the unit disk  $\mathbb{B}^2$  in the two-dimensional plane  $\mathbb{C} = \mathbb{R}^2$ . Naturally, we use here the Riemann mapping theorem [2]. After this preliminary reduction, we look for upper bounds for the condenser capacity  $\text{cap}(\mathbb{B}^2, E)$  in terms of the hyperbolic metric  $\rho_{\mathbb{B}^2}$  of the unit disk, when the hyperbolic diameter  $\rho_{\mathbb{B}^2}(E)$  of the compact set  $E$  is fixed.

A first guess might be that for a compact set  $E \subset \mathbb{B}^2$ , a majorant for  $\text{cap}(\mathbb{B}^2, E)$  would be the capacity of a hyperbolic disk with the hyperbolic diameter equal to that of  $E$ . This guess is motivated by a measure-theoretic isodiametric inequality, see Remark 3.14. However, the main result of this paper is to show that this guess is wrong. For this purpose, we introduce the so-called hyperbolic Reuleaux triangle, which is a set of constant hyperbolic width, and then compute its conformal capacity with novel computational methods [13,16,18,19] to confirm our claim. A valid upper bound for  $\text{cap}(\mathbb{B}^2, E)$  in terms of the hyperbolic diameter is instead naturally given in terms of the capacity of the minimal hyperbolic disk containing the set  $E$  and here we apply the work of B.V. Dekster in [4] who found this minimal radius.

**Theorem 1.3.** *For a continuum  $E \subset \mathbb{B}^2$  with the hyperbolic diameter equal to  $t > 0$ , the inequality*

$$\text{cap}(\mathbb{B}^2, E) \leq \frac{2\pi}{\log((1 + \sqrt{1 + v^2})/v)} \quad \text{with} \quad v = (2/\sqrt{3}) \text{sh}(t/2) \quad (1.4)$$

*holds.*

Given a number  $t > 0$ , the *hyperbolic Jung radius* is the smallest number  $r > 0$  such that every set  $E \subset \mathbb{B}^2$  with the hyperbolic diameter equal to  $t$  is contained in some hyperbolic disk with the radius equal to  $r$ . Originally, the Jung radius was found in the context of the Euclidean geometry [14, p. 33, Thm 2.8.4] and its hyperbolic counterpart was found for dimensions  $n \geq 2$  by B.V. Dekster in [4]. His result is formulated below as Theorem 3.3 and Theorem 1.3 is based on the special case  $n = 2$  of his work. It should be noticed that, by the Riemann mapping theorem, (1.4) directly applies to the case of planar simply connected domains. The sharp upper bound in Theorem 1.3 is not known and this motivates the following open problem.

**1.5. Open problem.** Given  $t > 0$ , identify all connected compact sets  $E \subset \mathbb{B}^2$  with the hyperbolic diameter  $t$ , which maximize the capacity  $\text{cap}(\mathbb{B}^2, E)$ .

Theorem 1.3 provides an upper bound for the quantity

$$b(t) \equiv \sup\{\text{cap}(\mathbb{B}^2, E) : E \text{ continuum and } \rho(E) = t\} \tag{1.6}$$

that we will analyze further, in order to find a lower bound for it. To this end we have to apply numerical methods. Our first step is to write an algorithm for computing the hyperbolic diameter of a set in a simply connected domain. The boundary integral equation method developed in a series of recent papers [13,16,18,19] is used. Using this method, we can compute the hyperbolic diameter and the capacity of a subset bounded by piecewise smooth curves in a polygonal domain or in the unit disk. We show that the capacity of a hyperbolic disk with diameter  $t$ , denoted  $b_1(t)$ , is a minorant for the above function  $b(t)$ , i.e.  $b(t) \geq b_1(t)$ , see (3.12). For this purpose we introduce the aforementioned hyperbolic Reuleaux triangle and our numerical work shows that its capacity majorizes the capacity of a disk with the same hyperbolic diameter. A delicate point here is the essential role of the hyperbolic geometry: the hyperbolic Reuleaux triangle cannot be replaced by the Euclidean Reuleaux triangle with the same hyperbolic diameter, for its capacity is not a majorant for  $b_1(t)$  for  $t > 2$ . The numerical algorithm is of independent interest, because it enables one to experimentally study the hyperbolic geometry of planar simply connected polygonal domains.

We apply our result to quasiconformal maps and prove the following result.

**Theorem 1.7.** Let  $f : G_1 \rightarrow G_2 = f(G_1)$  be a  $K$ -quasiconformal homeomorphism between two simply connected domains  $G_1$  and  $G_2$  in  $\mathbb{R}^2$ , and let  $E \subset G_1$  be a continuum. Then

$$\text{th} \frac{\rho_{G_2}(f(E))}{2} \leq 4 \left( \text{th} \frac{h(2, \rho_{G_1}(E))}{2} \right)^{1/K} \tag{1.8}$$

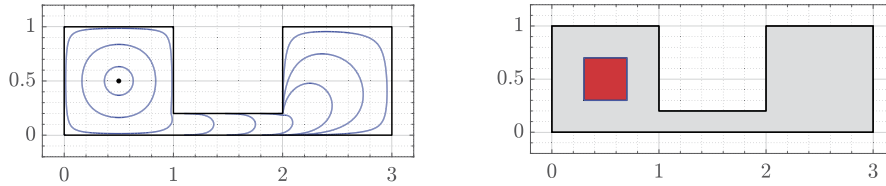
where  $\rho_{G_1}$  and  $\rho_{G_2}$  refer to the hyperbolic metrics of  $G_1$  and  $G_2$ , resp., and  $h(2, t)$  stands for the hyperbolic Jung radius of a set with the hyperbolic diameter equal to  $t$  defined in Theorem 3.3 due to B.V. Dekster [4].

For a large class of simply connected plane domains, so called  $\varphi$ -uniform domains, we give explicit bounds for the hyperbolic Jung radius of a compact set in the domain.

We are indebted to Prof. Alex Solynin for pointing out the above open problem.

## 2. Preliminaries

An open ball defined with the Euclidean metric is  $B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$  and the corresponding closed ball is  $\overline{B}^n(x, r) = \{y \in \mathbb{R}^n : |x - y| \leq r\}$ . The sphere of these balls is  $S^{n-1}(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\}$ . Note that if the center  $x$  or the radius  $r$  is not otherwise specified in these notations, it means that  $x = 0$  and  $r = 1$ . In a metric space  $(X, d)$ , a ball centered at  $x$  and with radius  $r > 0$  is  $B_d(x, r)$  and the diameter of a non-empty set  $A \subset X$  is  $d(A)$ .



**Fig. 1.** Examples of hyperbolic circles in a simply connected polygonal domain  $G$  (left) and a square with the vertices  $0.5 \pm h + (0.5 \pm h)i$  for  $h = 0.2$  in the simply connected polygonal domain  $G$  (right).

In the Poincaré unit ball  $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ , the hyperbolic metric is defined as [2, (2.8) p. 15]

$$\text{sh}^2 \frac{\rho_{\mathbb{B}^n}(x, y)}{2} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \quad x, y \in \mathbb{B}^n.$$

The hyperbolic segment between the points  $x, y$  is denoted by  $J[x, y]$ . Furthermore, the hyperbolic balls  $B_\rho(q, R)$  are Euclidean balls with the center and the radius given by the following lemma.

**Lemma 2.1.** ([9, (4.20) p. 56]) *The equality  $B_\rho(q, R) = B^n(j, h)$  holds for  $q \in \mathbb{B}^n$  and  $R > 0$ , if*

$$j = \frac{q(1 - t^2)}{1 - |q|^2 t^2}, \quad h = \frac{(1 - |q|^2)t}{1 - |q|^2 t^2} \quad \text{and} \quad t = \text{th}(R/2).$$

For a given simply connected planar domain  $G$ , by means of the Riemann mapping theorem, one can define a conformal map of  $G$  onto the unit disk  $\mathbb{B}^2$ ,  $f : G \rightarrow \mathbb{B}^2 = f(G)$ , and thus define the hyperbolic metric  $\rho_G$  in  $G$  by [2]

$$\rho_G(x, y) = \rho_{\mathbb{B}^2}(f(x), f(y)), \quad x, y \in G. \tag{2.2}$$

As an example, consider the simply connected domain  $G$  inside the polygon with the vertices  $0, 3, 3 + i, 2 + i, 2 + 0.2i, 1 + 0.2i, 1 + i$ , and  $i$ . Fig. 1 (left) displays examples of hyperbolic circles in the domain  $G$ . These hyperbolic circles are plotted by plotting the contour lines of the function

$$u(z) = \rho_G(\alpha, z), \quad z \in G,$$

corresponding to the levels (the hyperbolic radii of the hyperbolic circles)  $0.5, 1.5, 4, 10, 16, 21, 23, 23.5$ , and  $23.75$  where  $\alpha = 0.5 + 0.5i$ . The values of  $\rho_G(\alpha, z)$  are computed using the method described in Appendix A.1 with  $n = 2^{13}$ .

The hyperbolic diameter of a compact set  $E \subset G$ , denoted by  $\rho_G(E)$ , is defined by

$$\rho_G(E) = \sup\{\rho_G(x, y) \mid x, y \in E\}.$$

For the polygonal domain  $G$  in Fig. 1 (left), let  $E \subset G$  be the closure of the square with the vertices  $0.5 \pm h + (0.5 \pm h)i$  for  $0 < h < 0.5$  (see Fig. 1 (right) for  $h = 0.2$ ). The approximate values of the hyperbolic diameter of the set  $E$  for several values of  $h$ , computed by the method described in Appendix A.1 with  $\alpha = 0.5 + 0.5i$  and  $n = 2^{13}$ , are given in Table 1. Table 1 also presents the values of the capacity of the condenser  $(G, E)$ , which are computed using the method described in Appendix A.4 with  $\alpha = 1.5 + 0.1i$ ,  $z_2 = 0.5 + 0.5i$ , and  $n = 2^{13}$ .

Note that while we already defined the condenser capacity in (1.1), its definition can be also written as

$$\text{cap}(G, E) = M(\Delta(E, \partial G; G)),$$

**Table 1**  
The hyperbolic diameter  $\rho_G(E)$  and the capacity  $\text{cap}(G, E)$  for the sets  $G$  and  $E$  shown in Fig. 1 (right).

| $h$  | $\rho_G(E)$ | $\text{cap}(G, E)$ |
|------|-------------|--------------------|
| 0.1  | 1.0729      | 4.1331             |
| 0.2  | 2.3071      | 7.5564             |
| 0.3  | 3.9596      | 14.2096            |
| 0.4  | 6.7393      | 33.9643            |
| 0.45 | 9.5123      | 72.8330            |

as in [7, Thm 5.2.3, p. 164], [9, Thm 9.6, p. 152]. Here,  $\Delta(E, F; G)$  stands for the family of all the curves in the set  $G$  that have one end point in the set  $E$  and another end point in  $F$  [9, p. 106]. The definition and basic properties of the modulus  $M(\Gamma)$  of a curve family  $\Gamma$  can be found in [9, Ch. 7, pp. 103-131]. We often use the fact that the capacity is, in the same way as the modulus, conformally invariant.

**Lemma 2.3.** (1) If  $0 < a < b$  and  $D = \overline{B}^n(b) \setminus B^n(a)$ ,

$$M(\Delta(S^{n-1}(a), S^{n-1}(b); D)) = \omega_{n-1}(\log(b/a))^{1-n}.$$

(2) If  $R > 0$  then for  $x \in \mathbb{B}^n$  and  $R > 0$

$$M(\Delta(S^{n-1}, B_\rho(x, R); \mathbb{B}^n)) = \omega_{n-1}(\log(1/\text{th}(R/2)))^{1-n}.$$

Here,  $\omega_{n-1}$  is the  $(n - 1)$ -dimensional surface area of the unit sphere  $S^{n-1}$ . In particular,  $\omega_1 = 2\pi$ .

**Proof.** (1) This is a well-known basic fact, see e.g. [9, (7.3), p. 107].

(2) The value of the left hand side is independent of  $x$  by the Möbius invariance of the modulus and of the hyperbolic metric and hence we may assume that  $x = 0$ . By Lemma 2.1,  $B_\rho(x, R) = B_\rho(0, R) = B^n(0, \text{th}(R/2))$  and hence the proof follows from part (1).  $\square$

The Grötzsch and Teichmüller capacities are the following decreasing homeomorphisms  $\gamma_n : (1, \infty) \rightarrow (0, \infty)$ ,  $\tau_n : (0, \infty) \rightarrow (0, \infty)$  [9, (7.17), p. 121]:

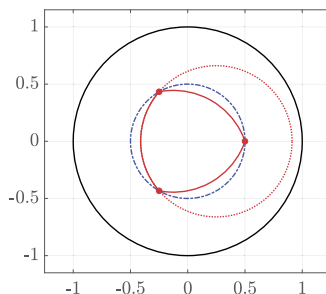
$$\begin{aligned} \gamma_n(s) &= M(\Delta(\overline{\mathbb{B}}^n, [se_1, \infty]; \mathbb{R}^n)), \quad s > 1, \\ \tau_n(s) &= M(\Delta([-e_1, 0], [se_1, \infty]; \mathbb{R}^n)), \quad s > 0, \end{aligned}$$

where the notation  $e_1, \dots, e_n$  stands for the unit vectors of  $\mathbb{R}^n$ . These capacities satisfy  $\gamma_n(s) = 2^{n-1}\tau_n(s^2-1)$ , for  $s > 1$  and various estimates are given in [9, Chapter 9] for  $n \geq 3$ . For  $n = 2, r \in (0, 1)$ , the following explicit formulas are given by [9, (7.18), p. 122],

$$\gamma_2(1/r) = \frac{2\pi}{\mu(r)}; \quad \mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}. \tag{2.4}$$

**Lemma 2.5.** (1) [9, Lemma 9.20, p. 163] If  $x, y \in \mathbb{B}^n, x \neq y$ , and  $E \subset \mathbb{B}^n$  is a continuum with  $x, y \in E$ , then

$$\text{cap}(\mathbb{B}^n, E) \geq \gamma_n \left( \frac{1}{\text{th}(\rho_{\mathbb{B}^n}(x, y)/2)} \right).$$



**Fig. 2.** A hyperbolic Reuleaux triangle with vertices on  $\{z : |z| = 0.5\}$ . The dotted circular arc is a part of the boundary of the disk  $B_\rho(0.5, M)$ , which is one of the three hyperbolic disks defining this triangle, and  $M$  is the hyperbolic distance between vertices of the triangle.

Here, the equality holds if  $E$  is the geodesic segment  $J[x, y]$  of the hyperbolic metric joining  $x$  and  $y$ .

(2) If  $G$  is a simply connected domain in  $\mathbb{R}^2$ ,  $E \subset G$  is a continuum, and  $x, y \in G, x \neq y$ , then

$$\text{cap}(G, E) \geq \gamma_2 \left( \frac{1}{\text{th}(\rho_G(x, y)/2)} \right).$$

**Proof.** (2) By the Riemann mapping theorem, we may assume without loss of generality that  $G = \mathbb{B}^2$  and hence the proof follows from part (1).  $\square$

**2.6. Sets of constant width [14].** Let  $E \subset \mathbb{R}^n$  be a compact set with diameter equal to  $t > 0$ . We say that  $E$  is a set of constant width if for every  $z \in \partial E$ ,

$$t = \sup\{|z - x| : x \in E\}.$$

**2.7. The Euclidean and hyperbolic Reuleaux triangle.** An example of a set of constant width is *the Reuleaux triangle*, the intersection of three closed disks with radii equal to  $t > 0$  and with centers at the vertices of an equilateral triangle having side lengths equal to  $t$ . We can define *the hyperbolic Reuleaux triangle* (Fig. 2), a subset of the unit disk  $\mathbb{B}^2$  in the same way. To be more explicit, consider the hyperbolic Reuleaux triangle with vertices at  $r, r \exp(2\pi i/3), r \exp(4\pi i/3)$ , and let

$$M = \rho_{\mathbb{B}^2}(r, r \exp(2\pi i/3)) = 2 \operatorname{arsh} \frac{r\sqrt{3}}{1-r^2}.$$

By Lemma 2.1,  $B_\rho(r, M) = B^2(y, h)$  where  $y$  and  $h$  are given by

$$y = \frac{(1-t^2)r}{1-r^2t^2}, \quad h = \frac{(1-r^2)t}{1-r^2t^2}, \quad t = \operatorname{th}(\operatorname{arsh} \frac{r\sqrt{3}}{1-r^2}).$$

Let  $D_1 = \overline{B^2}(y, h)$  and let  $D_2, D_3$  be the disks obtained from  $D_1$  by rotation around the origin with angles  $2\pi/3$  and  $4\pi/3$ , resp. Now, the hyperbolic Reuleaux triangle with vertices at the above points is  $D_1 \cap D_2 \cap D_3$ .

### 3. Capacity and Jung radius

For a compact subset  $E$  of a metric space  $X$ , the Jung radius is the least number  $r > 0$  such that, for some  $x \in X$ ,  $E$  is a subset of the closed ball centered at  $x$  with the radius  $r$  [14]. The metric space in our work will be the hyperbolic disk and we denote the hyperbolic Jung radius of the set  $E$  by  $r_{Jung}(E)$ . Clearly, it follows from the conformal invariance of the hyperbolic metric that the Jung radius is conformally

invariant. Because of the same reason, for every simply connected domain  $G \subsetneq \mathbb{R}^2$  and all compact sets  $E \subset G$ , there exists  $z \in G$  with

$$\text{cap}(G, E) \leq \text{cap}(G, \overline{B}_\rho(z, r_{\text{Jung}}(E))). \tag{3.1}$$

By Lemma 2.1,  $B_\rho(z, r_{\text{Jung}}(E))$  is conformally equivalent to  $B^2(0, \text{th}(r_{\text{Jung}}(E)/2))$  and thus it follows from (3.1) and Riemann’s mapping theorem that

$$\text{cap}(G, E) \leq \frac{2\pi}{\log(1/\text{th}(r_{\text{Jung}}(E)/2))}. \tag{3.2}$$

**Theorem 3.3.** (B.V. Dekster [4, Thm 2, (1.3)]) *If  $E \subset \mathbb{B}^n$  is a compact set with  $\rho_{\mathbb{B}^n}(E) \leq t$ , then*

$$r_{\text{Jung}}(E) \leq \text{arsh} \left( \sqrt{\frac{2n}{n+1}} \text{sh} \frac{t}{2} \right) \equiv h(n, t).$$

**Remark 3.4.** Making use of the identity

$$\text{th} \frac{M}{2} = \frac{\text{sh}M}{1 + \sqrt{1 + \text{sh}^2M}}, \quad M > 0,$$

we observe that

$$\text{th} \frac{h(n, t)}{2} = \frac{u \text{sh}(t/2)}{1 + \sqrt{1 + u^2 \text{sh}^2(t/2)}}, \quad u = \sqrt{2n/(n+1)}, \quad t > 0. \tag{3.5}$$

For Lemma 3.7, which provides bounds for the function  $h(n, t)$ , we first prove some preliminary results.

**Proposition 3.6.** (1) *For all  $k > 0$ , the function  $h : (0, \infty) \rightarrow \mathbb{R}$ ,  $h(x) = \text{sh}(kx)/x$ , is increasing.*  
 (2) *For all  $x \geq 1$ ,  $k > 0$ , the inequality  $x \text{sh}k \leq \text{sh}(kx)$  holds.*

**Proof.** (1) Writing  $f(x) = \text{sh}(kx)$  and  $g(x) = x$ , we see that  $f'(x)/g'(x) = k \text{ch}(kx)$  is increasing and, by [9, Thm B.2, p. 465], so is  $h(x) = f(x)/g(x)$ .

(2) Since the function  $h(x) = \text{sh}(kx)/x$  of part (1) is increasing for all  $k > 0$ ,

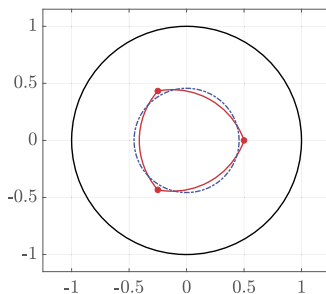
$$x \geq 1 \quad \Leftrightarrow \quad h(1) \leq h(x) \quad \Leftrightarrow \quad \text{sh}(k) \leq \text{sh}(kx)/x \quad \Leftrightarrow \quad x \text{sh}k \leq \text{sh}(kx). \quad \square$$

**Lemma 3.7.** *For all  $n \geq 2$ ,  $t > 0$ , the inequality  $\sqrt{2(n+1)/n} \leq t/h(n, t) \leq 2$  holds.*

**Proof.** We can write

$$\begin{aligned} \sqrt{\frac{2(n+1)}{n}} \leq t/h(n, t) &\Leftrightarrow h(n, t) = \text{arsh} \left( \sqrt{\frac{2n}{n+1}} \text{sh} \frac{t}{2} \right) \leq t \sqrt{\frac{n}{2(n+1)}} \\ &\Leftrightarrow \sqrt{\frac{2n}{n+1}} \text{sh} \frac{t}{2} \leq \text{sh} \left( \frac{t}{2} \sqrt{\frac{2n}{n+1}} \right). \end{aligned}$$

By choosing  $x = \sqrt{2n/(n+1)} \geq 2/\sqrt{3} > 1$  and  $k = t/2 > 0$ , we see that the inequality follows from Proposition 3.6. Furthermore, since



**Fig. 3.** A hyperbolic Reuleaux triangle  $T$  with vertices on  $\{z : |z| = 0.5\}$  and a hyperbolic disk  $D$  with the hyperbolic diameter equal to that of  $T$ . Note that  $\text{cap}(\mathbb{B}^2, D) < \text{cap}(\mathbb{B}^2, T)$ .

$$\begin{aligned} t/h(n, t) \leq 2 &\Leftrightarrow h(n, t) = \text{arsh} \left( \sqrt{\frac{2n}{n+1}} \text{sh} \frac{t}{2} \right) \geq \frac{t}{2} \Leftrightarrow \sqrt{\frac{2n}{n+1}} \text{sh} \frac{t}{2} \geq \text{sh} \frac{t}{2} \\ &\Leftrightarrow \sqrt{\frac{2n}{n+1}} \geq 1 \Leftrightarrow n \geq 1, \end{aligned}$$

the latter inequality in the lemma also holds.  $\square$

**Corollary 3.8.** (1) If  $E$  is a compact subset of the unit ball  $\mathbb{B}^n$ ,  $n \geq 2$ , with the hyperbolic diameter at most  $t$ , then

$$\text{cap}(\mathbb{B}^n, E) \leq \frac{\omega_{n-1}}{(\log(1/\text{th}(h(n, t)/2)))^{n-1}}, \quad t \leq 2h(n, t) \leq t\sqrt{2n/(n+1)}.$$

(2) If  $E$  is a compact subset of a simply connected domain  $G \subseteq \mathbb{R}^2$ , then

$$\text{cap}(G, E) \leq \frac{2\pi}{\log((1 + \sqrt{1+v^2})/v)} \quad \text{with} \quad v = \sqrt{\frac{4}{3}} \text{sh} \frac{\rho_G(E)}{2}.$$

**Proof.** (1) follows immediately from Theorem 3.3, Lemma 3.7 and Lemma 2.3 and some basic properties of the modulus.

(2) The proof follows from (3.2) and the identity (3.5).  $\square$

**3.9. Proof of Theorem 1.3.** The proof follows from Corollary 3.8(2).  $\square$

**Remark 3.10.** We compare here the capacities of several sets in terms of the hyperbolic diameter  $t$ . The results are parametrized so that the vertices of the Reuleaux triangle are on the circle  $|z| = r$ . The results are given in the following table organized in seven columns as follows: (1)  $r = \text{th}(t/2)$ , (2)  $t$ , (3)  $2\pi/\mu(\text{th}(t/2))$  i.e. the capacity of the hyperbolic geodesic segment of diameter  $t$ , (4) the capacity of a Euclidean Reuleaux triangle with hyperbolic diameter  $t$ , (5)  $2\pi/\log(1/\text{th}(t/4))$  i.e. the capacity of a hyperbolic disk with diameter  $t$ , (6) the capacity of a hyperbolic Reuleaux triangle with diameter  $t$ , (7) the upper bound given by Corollary 3.8. The values in columns (4) and (6) are computed using the method described in Appendix A.4 with  $\alpha = 0.4 + 0.6r$ ,  $z_2 = 0$ , and  $n = 3 \times 2^8$ . (See Fig. 3.)

**3.11. Comparison of the bounds.** Corollary 3.8(2) gives an upper bound  $b_2(t)$  for

$$b(t) \equiv \sup\{\text{cap}(\mathbb{B}^2, E) : E \text{ connected and } \rho(E) = t\}.$$

As we have seen above,



**Table 2**  
The values of the computational results in Remark 3.10.

| $r$  | h-diam | capSeg  | capERtri | capDisk  | capHRtri | capJung  |
|------|--------|---------|----------|----------|----------|----------|
| 0.05 | 0.1734 | 1.6396  | 2.0242   | 2.0017   | 2.0245   | 2.0974   |
| 0.15 | 0.5255 | 2.3028  | 3.1332   | 3.0869   | 3.1397   | 3.3120   |
| 0.25 | 0.8937 | 2.8457  | 4.2040   | 4.1470   | 4.2360   | 4.5324   |
| 0.35 | 1.2903 | 3.3831  | 5.4255   | 5.3920   | 5.5272   | 5.9850   |
| 0.45 | 1.7305 | 3.9583  | 6.9289   | 6.9994   | 7.1957   | 7.8687   |
| 0.55 | 2.2359 | 4.6082  | 8.8968   | 9.2558   | 9.5369   | 10.5099  |
| 0.65 | 2.8416 | 5.3821  | 11.6482  | 12.7508  | 13.1588  | 14.5855  |
| 0.75 | 3.6173 | 6.3706  | 15.8319  | 18.9982  | 19.6196  | 21.8407  |
| 0.85 | 4.7413 | 7.8018  | 23.0155  | 33.5301  | 34.5948  | 38.6613  |
| 0.95 | 7.0399 | 10.7285 | 38.0667  | 106.0995 | 108.9365 | 122.4953 |

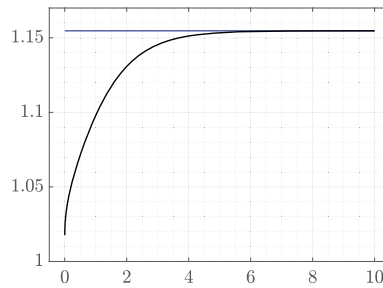


Fig. 4. Graph of the quotient  $b_2(t)/b_1(t)$  and  $2/\sqrt{3}$ .

$$b(t) \geq 2\pi/\log(1/\text{th}(t/4)) \equiv b_1(t).$$

Define  $b_2(t)$  as in Corollary 3.8(2). Now, we know that for a hyperbolic Reuleaux triangle  $T$  of hyperbolic diameter equal to  $t$  we have

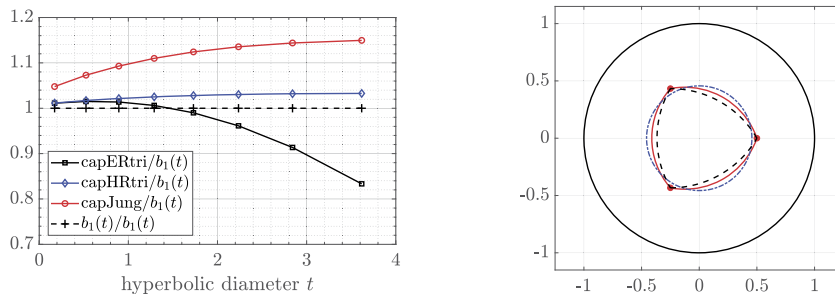
$$b_1(t) \leq \text{cap}(\mathbb{B}^2, T) \leq b_2(t). \tag{3.12}$$

Fig. 4 displays the graph of the function  $b_2(t)/b_1(t)$  and its limit value  $2/\sqrt{3}$  when  $t \rightarrow \infty$ .

**3.13. Capacity comparison: the Euclidean vs hyperbolic Reuleaux triangle.** We have shown above that the hyperbolic Reuleaux triangle of diameter  $t$  has a larger capacity than  $b_1(t)$ , the capacity of a hyperbolic disk with the same diameter. A natural question is: Why do we use for this purpose the hyperbolic Reuleaux triangle, not the Euclidean one? It follows easily from Lemma 2.1 that, as a point set, the hyperbolic triangle contains the Euclidean one and thus has a larger capacity. The key point now is that the capacity of the Euclidean Reuleaux triangle is smaller than  $b_1(t)$  for  $t > 2$ . In Fig. 5 (left) we demonstrate this fact by graphing, as a function of  $t$ , the four quotients (1) Jung bound Corollary 3.8(2) divided by  $b_1(t)$ , (2) the capacity of the hyperbolic Reuleaux triangle/ $b_1(t)$ , (3)  $b_1(t)/b_1(t)$  (horizontal line), (4) the capacity of the Euclidean Reuleaux triangle/ $b_1(t)$ .

Fig. 5 (right) displays three sets of equal hyperbolic diameter: a disk, a hyperbolic Reuleaux triangle (solid line) and a Euclidean Reuleaux triangle (dashed line).

**Remark 3.14.** For the Lebesgue measure of a measurable set  $E \subset \mathbb{R}^n$ , the well-known isodiametric inequality states that  $m(E) \leq m(B^n(0, r))$  where the Euclidean diameter of  $E$  is  $2r$  [12, p. 548, Thm C.10]. A similar result was proven very recently by K.J. Böröczky and Á. Sagmeister in [3] for the balls in the hyperbolic geometry. As the above computational results demonstrate, for the condenser capacity there is no similar result.



**Fig. 5.** On the left, the capacities divided by  $b_1(t)$  (see (3.12)) as a function of the hyperbolic diameter  $t$ . On the right, three sets of equal hyperbolic diameter: a disk  $D$  (dash-dotted line), a hyperbolic Reuleaux triangle  $T$  (solid line), and a Euclidean Reuleaux triangle  $E$  (dashed line),  $E \subset T$ . Note that by the results in Table 2 we have  $\text{cap}(\mathbb{B}^2, E) < \text{cap}(\mathbb{B}^2, D) < \text{cap}(\mathbb{B}^2, T)$  for all large enough  $t$ .

**3.15. Proof of Theorem 1.7.** Due to the conformal invariance of the hyperbolic metric, we may assume without loss of generality that  $G_1 = G_2 = \mathbb{B}^2$ . Let  $\Delta = \Delta(E, \partial\mathbb{B}^2; \mathbb{B}^2)$  be the family of all curves in  $\mathbb{B}^2$  joining  $E$  and  $\partial\mathbb{B}^2$ . By quasiconformality,

$$M(f\Delta) \leq KM(\Delta). \tag{3.16}$$

Next, because  $\mu(t) < \log(4/r)$  by [9, (7.21)] for  $r \in (0, 1)$ , we obtain by Lemma 2.5(2) and (2.4) that

$$M(f\Delta) \geq \gamma_2 \left( \frac{1}{\text{th}(\rho_{G_2}(f(E))/2)} \right) \geq \frac{2\pi}{\mu(\text{th}(\rho_{G_2}(f(E))/2))} \geq \frac{2\pi}{\log(4/(\text{th}(\rho_{G_2}(f(E))/2)))}. \tag{3.17}$$

On the other hand, by Corollary 3.8,

$$M(f\Delta) \leq \frac{2\pi}{\log(1/(\text{th}(h(2, \rho_{G_1}(E))/2)))}. \tag{3.18}$$

The inequalities (3.16), (3.17) and (3.18) together yield

$$\text{th} \frac{\rho_{G_2}(f(E))}{2} \leq 4 \left( \text{th} \frac{h(2, \rho_{G_1}(E))}{2} \right)^{1/K},$$

as desired.  $\square$

#### 4. Upper bounds for the hyperbolic Jung radius

In view of Corollary 3.8, it is natural to look for bounds of the hyperbolic Jung radius of a compact set in a simply connected plane domain  $G$ . Perhaps a first question to study is whether we can find an upper bound in terms of the domain functional  $d(E)/d(E, \partial G)$ . As Example 4.1 demonstrates, this is not true in general simply connected domains, but by (4.4) such a majorant is valid for  $\varphi$ -uniform domains.

**Example 4.1.** For  $G = \mathbb{B}^2 \setminus [0, 1)$ , let  $t \in (0, 1/4)$ , fix the points  $x_t = (1/2, t)$ ,  $y_t = (1/2, -t)$ , and let  $E_t$  be the set  $\{x_t, y_t\} \subset G$ . Then  $d(E_t)/d(E_t, \partial G) = 2$  but  $\rho_G(x_t, y_t) \rightarrow \infty$  if  $t \rightarrow 0^+$ . Therefore, the hyperbolic Jung radius has no bounds in terms of  $d(E)/d(E, \partial G)$ .

**4.2.  $\varphi$ -uniform domains.** Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be an increasing homeomorphism and  $G \subset \mathbb{R}^2$  a simply connected domain. We say that  $G$  is  $\varphi$ -uniform if

$$\rho_G(x, y) \leq \varphi \left( \frac{|x - y|}{\min\{d_G(x), d_G(y)\}} \right) \tag{4.3}$$

for all  $x, y \in G$ .

The class of  $\varphi$ -uniform domains [9, pp. 84-85] contains many types of domains, including, for instance, all convex domains and so called quasidisks, which are images of the unit disk under quasiconformal maps of the plane [6].

Now, we observe that if  $E$  is a compact subset of a simply connected  $\varphi$ -uniform domain  $G$ , then by Theorem 3.3,

$$r_{Jung}(E) \leq \operatorname{arsh} \left( \frac{2}{\sqrt{3}} \operatorname{sh} \frac{\varphi(d(E)/d(E, \partial G))}{2} \right). \tag{4.4}$$

Finally, we give a simple sufficient condition for a domain  $G \subset \mathbb{R}^2$  to be  $\varphi$ -uniform: There exists  $c \geq 1$  such that every pair of points  $x, y$  in  $G$  can be joined by a curve  $\gamma$  with length at most  $c|x - y|$  so that

$$d(\gamma, \partial G) \geq (1/c) \min\{d_G(x), d_G(y)\}.$$

For more details, see [6, p. 35] and [9, pp. 84-85].

**Remark 4.5.** Recall that in every plane domain  $G$ , the hyperbolic diameter of a continuum  $E \subset G$  is bounded in terms of  $d(E)/d(E, \partial G)$  [9, 6.32] and hence so is its hyperbolic Jung radius by Theorem 3.3.

**Data availability**

All the data used in the research for this article was created with MATLAB codes available in GitHub at [github.com/mmsnasser/hypdiam](https://github.com/mmsnasser/hypdiam).

**Appendix A. Computational methods**

Computational tools for computing several conformal invariants in simply and doubly connected domains have been presented recently in [18,19]. These tools are based on using the boundary integral equation with the generalized Neumann kernel. A fast numerical method for solving the integral equation is presented in [16] which makes use of the Fast Multipole Method toolbox [8]. In this appendix, we briefly describe these tools and demonstrate how they can be applied to compute numerically the hyperbolic diameter of compact sets as well as the capacity of condensers.

**A.1. Numerical computation of hyperbolic diameter.** If  $E \subset G$  is a compact set in a simply connected domain  $G$ , then the hyperbolic diameter of  $E$  with respect to  $G$ ,

$$\rho_G(E) = \max\{\rho_G(x, y) \mid x, y \in \partial E\}, \tag{A.2}$$

can be computed once we have an algorithm for the hyperbolic distance  $\rho_G(x, y)$ . In this paper, the maximum in (A.2) is approximated numerically by discretizing the boundary  $\partial E$  with a sufficiently large number of points. The hyperbolic distance  $\rho_G(x, y)$  itself is approximated numerically by approximating a conformal mapping from the domain  $G$  onto the unit disk  $\mathbb{B}^2$ .

A MATLAB function `hypdist` for approximating the hyperbolic distance  $\rho_G(x, y)$  when the boundary  $\Gamma = \partial G$  is a piecewise smooth Jordan curve is presented in [18]. To use the function `hypdist`, we parametrize  $\Gamma$ , which is assumed to be oriented counterclockwise, by a  $2\pi$ -periodic complex function  $\eta(\delta(t))$ ,  $t \in [0, 2\pi]$ , where  $\delta : [0, 2\pi] \rightarrow [0, 2\pi]$  is a bijective strictly monotonically increasing function. When  $\Gamma$  is smooth, we

choose  $\delta(t) = 1$ . For piecewise smooth boundary  $\Gamma$ , the function  $\delta$  is chosen as described in [13, p. 697] (see also [11]). We define  $n$  equidistant nodes  $s_1, \dots, s_n$  in the interval  $[0, 2\pi]$  by

$$s_k = (k-1) \frac{2\pi}{n}, \quad k = 1, \dots, n, \quad (\text{A.3})$$

where  $n$  is an even integer. Then we compute the vectors `et` and `etp` by

$$\text{et} = \eta(\delta(\mathbf{s})) \in \mathbb{C}^n, \quad \text{etp} = \eta'(\delta(\mathbf{s}))\delta'(\mathbf{s}) \in \mathbb{C}^n,$$

where  $\mathbf{s} = [s_1, \dots, s_n] \in \mathbb{R}^n$ . We also discretize the boundary of  $E$  by a vector of points  $\mathbf{z}$ . Then, the hyperbolic diameter  $\rho_G(E)$  is approximated by calling

$$\text{max}(\text{max}(\text{hypdist}(\text{et}, \text{etp}, n, \alpha, \mathbf{z}, \mathbf{z}))),$$

where  $\alpha$  is an auxiliary point in  $G$ . For more details, we refer the reader to [18].

**A.4. Numerical computation of the capacity.** Consider a bounded simply connected domain  $G$  in the plane and a compact set  $E \subset G$  such that  $D = G \setminus E$  is a doubly connected domain. In this paper, the capacity of the condenser  $(G, E)$  will be computed by the MATLAB function `annq` from [19]. The boundary components of  $D = G \setminus E$  are assumed to be piecewise smooth Jordan curves. Let  $\Gamma_1$  be the external boundary component and  $\Gamma_2$  be the inner boundary component such that  $\Gamma_1$  is oriented counterclockwise and  $\Gamma_2$  is oriented clockwise. We parametrize  $\Gamma_j$  by a  $2\pi$ -periodic complex function  $\eta_j(\delta_j(t))$ ,  $t \in [0, 2\pi]$ , where  $\delta_j : [0, 2\pi] \rightarrow [0, 2\pi]$  is a bijective strictly monotonically increasing function,  $j = 1, 2$ . When  $\Gamma_j$  is smooth, we choose  $\delta_j(t) = 1$ . For piecewise smooth boundary component  $\Gamma_j$ , the function  $\delta_j$  is chosen as in [13, p. 697]. We compute the vectors `et` and `etp` by

$$\begin{aligned} \text{et} &= [\eta_1(\delta_1(\mathbf{s})), \eta_2(\delta_2(\mathbf{s}))] \in \mathbb{C}^{2n}, \\ \text{etp} &= [\eta_1'(\delta_1(\mathbf{s}))\delta_1'(\mathbf{s}), \eta_2'(\delta_2(\mathbf{s}))\delta_2'(\mathbf{s})] \in \mathbb{C}^{2n}, \end{aligned}$$

where  $\mathbf{s} = [s_1, \dots, s_n] \in \mathbb{R}^n$  and  $s_1, \dots, s_n$  are given by (A.3). Then the MATLAB function `annq` can be used to approximate  $\text{cap}(G, E)$  as follows,

$$[\sim, \text{cap}] = \text{annq}(\text{et}, \text{etp}, n, \alpha, \mathbf{z}_2, \text{'b'}),$$

where  $\alpha$  is an auxiliary point in the domain  $D$  and  $\mathbf{z}_2$  is an auxiliary point in the interior of  $E$  (see Fig. 1 (right)). The readers are referred to [19] for more details.

The values of the parameters in the functions `hypdist` and `annq` are chosen as in [18, 19]. The codes for all presented computations in this paper are available in the link <https://github.com/mmsnasser/hypdiam>.

## References

- [1] A. Baernstein, *Symmetrization in Analysis*, with David Drasin and Richard S. Laugesen, with a foreword by Walter Hayman, New Mathematical Monographs, vol. 36, Cambridge University Press, Cambridge, 2019, xviii+473 pp.
- [2] A.F. Beardon, D. Minda, *The hyperbolic metric and geometric function theory*, in: S. Ponnusamy, T. Sugawa, M. Vuorinen (Eds.), Proc. International Workshop on Quasiconformal Mappings and Their Applications (IWQCMA05), 2006, pp. 9–56.
- [3] K.J. Böröczky, Á. Sagmeister, *The isodiametric problem on the sphere and in the hyperbolic space*, Acta Math. Hung. 160 (1) (2020) 13–32.
- [4] B.V. Dekster, *The Jung theorem for spherical and hyperbolic spaces*, Acta Math. Hung. 67 (4) (1995) 315–331.
- [5] V.N. Dubinin, *Condenser Capacities and Symmetrization in Geometric Function Theory*, Birkhäuser, 2014.
- [6] F.W. Gehring, K. Hag, *The Ubiquitous Quasidisk*, Mathematical Surveys and Monographs, vol. 184, American Mathematical Society, Providence, RI, 2012, with contributions by Ole Jacob Broch.

- [7] F.W. Gehring, G.J. Martin, B.P. Palka, *An Introduction to the Theory of Higher-Dimensional Quasiconformal Mappings*, Mathematical Surveys and Monographs, vol. 216, American Mathematical Society, Providence, RI, 2017, ix+430 pp.
- [8] L. Greengard, Z. Gimbutas, FMMLIB2D: a MATLAB toolbox for fast multipole method in two dimensions, version 1.2, [www.cims.nyu.edu/cmcl/fmm2dlib/fmm2dlib.html](http://www.cims.nyu.edu/cmcl/fmm2dlib/fmm2dlib.html), 2019. (Accessed 6 November 2020).
- [9] P. Hariri, R. Klén, M. Vuorinen, *Conformally Invariant Metrics and Quasiconformal Mappings*, Springer Monographs in Mathematics, Springer, Berlin, 2020.
- [10] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993; Revised and extended edition: Dover, 2006.
- [11] R. Kress, A Nyström method for boundary integral equations in domains with corners, *Numer. Math.* 58 (2) (1990) 145–161.
- [12] G. Leoni, *A First Course in Sobolev Spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009, xvi+607 pp.
- [13] J. Liesen, O. Sète, M.M.S. Nasser, Fast and accurate computation of the logarithmic capacity of compact sets, *Comput. Methods Funct. Theory* 17 (2017) 689–713.
- [14] H. Martini, L. Montejano, D. Oliveros, *Bodies of Constant Width. An Introduction to Convex Geometry with Applications*, Birkhäuser/Springer, Cham, 2019, xi+486 pp.
- [15] V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, second, revised and augmented edition, *Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences)*, vol. 342, Springer, Heidelberg, 2011, xxviii+866 pp.
- [16] M.M.S. Nasser, Fast solution of boundary integral equations with the generalized Neumann kernel, *Electron. Trans. Numer. Anal.* 44 (2015) 189–229.
- [17] M.M.S. Nasser, O. Rainio, M. Vuorinen, Condenser capacity and hyperbolic perimeter, *Comput. Math. Appl.* (2021), to appear.
- [18] M.M.S. Nasser, M. Vuorinen, Conformal invariants in simply connected domains, *Comput. Methods Funct. Theory* 20 (2020) 747–775.
- [19] M.M.S. Nasser, M. Vuorinen, Computation of conformal invariants, *Appl. Math. Comput.* 389 (2021) 125617.
- [20] G. Pólya, G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, *Annals of Mathematics Studies*, vol. 27, Princeton University Press, Princeton, NJ, 1951, xvi+279 pp.
- [21] Yu.G. Reshetnyak, *Space Mappings with Bounded Distortion*, translated from the Russian by H.H. McFaden, *Translations of Mathematical Monographs*, vol. 73, American Mathematical Society, Providence, RI, 1989, xvi+362 pp.
- [22] S. Rickman, *Quasiregular Mappings*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3))*, vol. 26, Springer-Verlag, Berlin, 1993, x+213 pp.