Engineering Journal of Qatar University, Vol. 5, 1992, p. 69 - 85

THREE DIMENSIONAL ANALYSIS OF SKELETAL STRUCTURES CONSIDERING CRACKING

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(First Received May 1992)

ABSTRACT

A three dimensional computer model has been developed to analyze space structures containing cracked members. A new stiffness matrix for cracked space members has been obtained. This model takes into consideration crack surface mixed mode displacements due to the presence of axial and shearing forces and bending and torsional moments. Based on the present analysis, the redistribution of internal forces and moments and deformations in structural members can be obtained taking into consideration the effect of crack size and location and geometry of the structure. Examples of place frame, plane grid and space frame have been analyzed using a computer program developed based on the present model. This analysis can be employed to identify overstressed regions in plane or space structures due to members cracking.

NOMENCLATURE

A	Cracked surface area
A_x	Cross sectional area
a	Crack depth, F.g. 1
B	Thickness of member cross section, Fig. 1
b	Distance from crack front to torsional force T, Fig. 3
d	Depth of member cross section, Fig. 2
E	Young's modulus
F	Applied force
G	Shear modulus
I	Moment of inertia of the cross section
J	Torsional rigidity of cross section
K	Stress intensity factor
L	Length of the cracked member
L_1	Distance of crack measured from left end, Fig. 1
L_2	Distance of crack measured from right end, Fig. 1
M	Bending moment, Fig. 2
M_t	Torsional moment, Fig. 3
n	Crack depth ratio (n = a/d), Fig. 5 - Fig. 7
P	Axial force, Fig. 2
V	Shearing force, Fig. 2

υ	Poission's ratio
δ	Horizontal deflection of frame
τ_{xy}	Shearing stress, Fig. 3
$\propto$	Factor depends on d/B
$\triangle\lambda_{ij}$	Displacement in direction i due to unit force applied in direction j
(F)	Flexibility matrix
(K)	Stiffness matrix
(S)	Stiffness matrix

INTRODUCTION

Cracking of structural members may take place due to severe loadings, aggressive environmental attacks, design and construction erros (1). An advanced structural analysis is needed to determine effects of cracks on the redistribution of internal forces and variations in structure deformations and hence evaluate the structural integrity and identify the overstressed regions in the structure due to the presence of these cracks. The finite element method is used in many applications to analyze cracked structures by choosing special crack tip elements around the crack tip and the cracked member is usually analyzed as a continuum. This approach is useful in analyzing plate or shell types of structures. However in skeletal structures, this method requires a large number of computaions. Instead, another approach based on the matrix method has been employed recently to analyze plane frames and beams (2-5). In this approach the cracked member is considered as a discrete element connected at its end nodes. The force displacement relations at these ends have been formulated based on fracture mechanics techniques where stiffness matrices corresponding to plane cracked members have been obtained (2). Effects of axial, bending and shear deformations due to crack presence were only considered in the above mentioned work which is only suitable for plane structures (2-5).

In the present paper, a new stiffness matrix has been developed to analyze three dimensional cracked structures taking into consideration effects of mixed mode displacements due to the action of axial and shearing forces, and bending and torsional moments which exist in local space members.

STRUCTURAL ANALYSIS MODEL OF CRACKED SPACE MEMBER

The relationship between displacements and forces at the two ends of the space frame member are assembled in a 12×12 matrix. To extend this method to analyze space structures containing cracked members as shown in Fig. 1a, the relation between the loads and the deformations at the two end nodes of that member in the form of a 12×12 matrix should be known. To obtain this matrix, the cracked member is divided into three elements as shown in Fig. 1b. The first and third elements are the common space frame elements with lengths L_1 and L_2 respectively. The second element of zero length simulates the cracked section and connects the first and third elements. This cracked member has four nodes and twenty two degrees of freedom, six at each node. This procedure has been employed recently (2) to obtain the stiffness matrix for cracked plane frame members, where effects of axial, shear and bending deformations due to crack presence were only considered. In the next section, this analysis is extended to analyze space structures by considering the combined effects of axial, shear, bending and torsion deformations due to crack presence. However effects of in plane shear deformations due to shearing forces, V and torsional moment, M_t

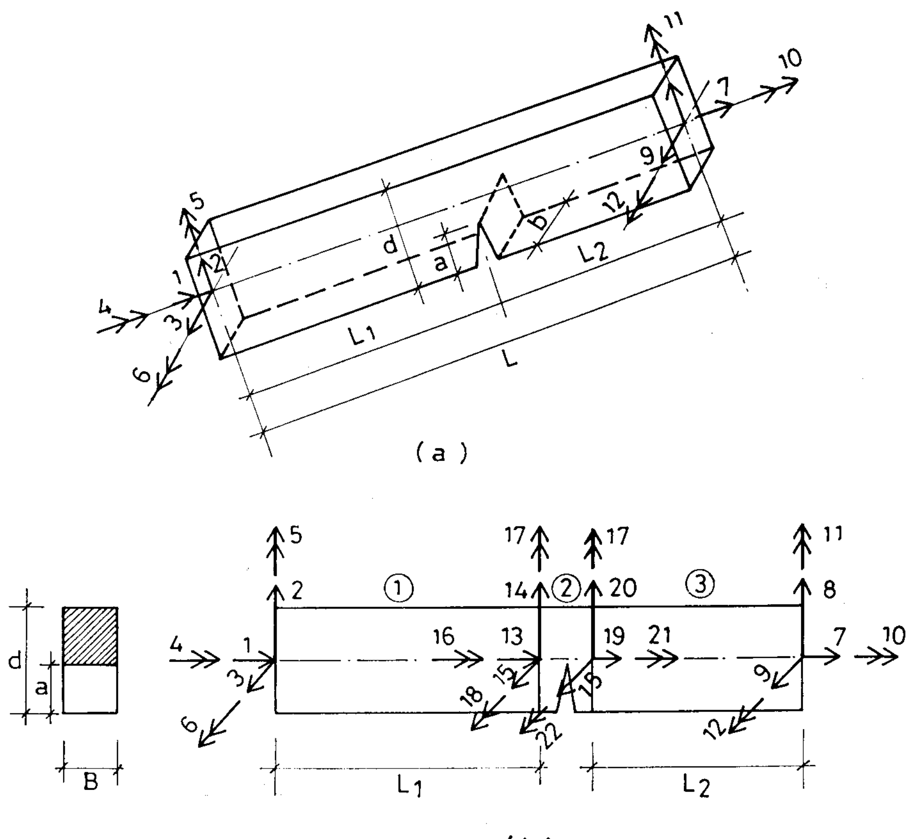


Fig. 1: Modeling of Cracked Space Member. (a) Geometry and degrees of freedom at the ends of cracked Member. (b) The Three elements represent the cracked space member.

shown in Fig. 2 are only considered. Effects of crack on out of plane shear deformations is neglected by keeping the same numbers 15 and 17 at both ends of cracked section corresponding to out of plane shearing and twisting deformations respectively.

Stiffness matrices of size 12 × 12 corresponding to the first and third space frame elements shown in Fig. 1b are known (6). These matrices are not valid for the second cracked element and therefore a new stiffness matrix simulating the relations between the forces and deformations at the ends of the cracked section is developed in the next section. This stiffness matrix together with the stiffness matrices representing the first and the third uncracked elements, will be used to obtain an overall stiffness matrix of the cracked member of size 22 × 22.

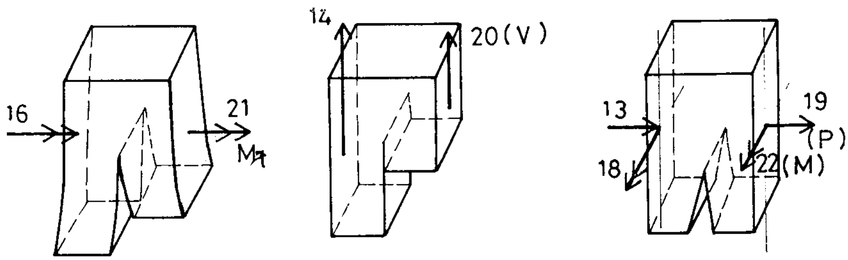
The system of numbering of the degrees of freedom at the four nodes shown in Fig. 1b is taken such that the degrees of freedom at the two intermediate nodes, at the ends of the cracked section, have the last numbers. This will be useful in the condensation of these nodes in order to obtain a stiffness matrix of size 12 × 12 relating the forces and displacements at the ends of this cracked space member.

STIFFNESS MATRIX OF CRACKED SECTION

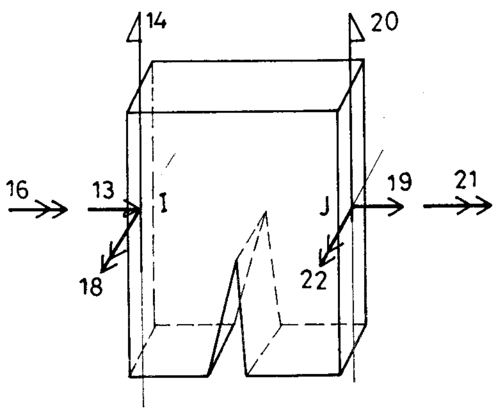
Stress Intensity Factor Solutions

In the present case of three dimensionals cracked member shown in Fig. 1, three modes of crack surface displacements are considered as shown in Fig. 2a. In mode I, crack surfaces are pulled apart due to the presence of axial force and bending moment leading to axial deformations in the direction of degrees of freedom 13 and 19 and rotations corresponding to the degrees of freedom 18 and 22. There are two other independent (shear) modes, as shown in Fig. 2a. Mode II is the in-plane forward mode, in which the crack surfaces moves normal to the crack tip but remain in the initial crack plane due to the application of shearing force V. The corresponding shear deformations act along the degrees of freedom 14 and 20 shown in the figure. Mode III is the anti-plane (parallel) shear mode and it is equivalent to a tearing motion due to the presence of torsional moment applied at the ends of cracked surfaces. The corresponding deformations is twisting angles in the direction of degrees of freedom 16 and 21 shown in Fig. 2a. Solutions for the stress intensity factors corresponding to the above three modes are given below which needed in order to determine the mixed modes of the crack surface displacements shown in Fig. 2b.

Solutions for the opening mode stress intensity factors for rectangular section subjects to axial force, P, and bending moment, M, and containing through crack shown in Fig. 2a are given below (7):



(a)



(b)

Fig. 2: Modeling of Cracked Section. (a) Three modes of crack surface displacements. Mode I (K_I) due to axial force (P) and bending moment M, Mode II (K_{II}) due to shearing force (V) and Mode III (K_{III}) due to torsional moment (M_t). (b) Mixed modes of crack surface displacements due to P, M, V and M_t.

$$K_{IP} = [P\sqrt{a}/(Bd)]\,[1.99 - 0.41n + 18.70n^2 - 38.48n^3 + 53.84n^4] \quad (1)$$

$$K_{IM} = [6M\sqrt{a}/(Bd^2)]\,[1.99 - 2.47n + 12.97n^2 - 23.17n^3 + 24.80n^4] \quad (2)$$

Solution for the stress intensity factor K_{II} corresponding the shear mode has been obtained recently by the author (2) and given below:

$$K_{IIV} = [V\sqrt{a}/(Bd)]\,[1.99 + 2.3n^{3}] \tag{3}$$

Solution for the stress intensity factor K_{IIIt} corresponding to torsional moment M_t is developed based on the solution of cracked plate subjected to a lateral force, T, applied at the crack surface at distance b measured from the crack edge as shown in Fig. 3, where K_{IIIt} is expressed at follows (7):

$$K_{IIIt} = (2/\sqrt{\pi a}).T/[B\sqrt{1-(b/a)^{2}}]$$

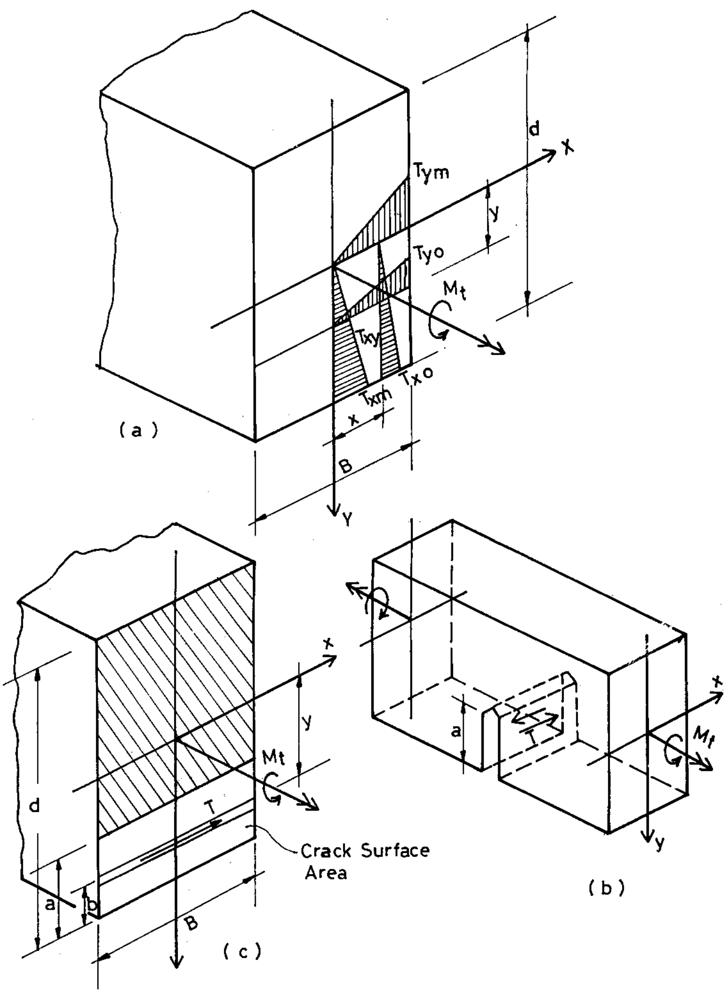


Fig. 3: Determination of Stress Intensity Factor due to Torsion. (a) Shear stress distribution due to torsion. (b) Crack in beam subjected to torsional moment. (c) Torsional force acting on cracked section.

The above expression is extended to the present case by assuming the shear stress distribution shown in Fig. 3a due to torsional moment M_t. As the shear stress distribution varies along x and y axes, Fig. 3a, a double integration is carried out to apply the above equation as follows:

$$K_{IIIt} = \int_{(d/2-a)}^{d/2} \int_{-B/2}^{B/2} (2/\sqrt{\pi a})\, \tau_{xy}\; dxdy / \left[B\sqrt{1-(d/2-y)^2/a^2}\right]$$

where τ_{xy} is given by the following expression (8):

$$\tau_{xy} = \tau_{xm}\,[1-(2x/B)^2]\; 2Y/d$$

and,

$$\tau_{xm} = M_t/[\alpha dB^2]$$

Where α is a factor depends on the d/B ratio (8) and is approximately is taken equal to 0.20.

Completing the above integration, an expression for K_{III} is obtained as follows:

$$K_{IIIt} = 1.9 M_t \sqrt{a}\; [\pi d - 4a]/(d/B)^2 \tag{4}$$

Crack Surface Displacements

Based on Castiglianos theorem and suggestions made by Paris (7), crack surface displacements in directions of the degrees of freedom due to mixed modes of loadings shown in Fig. 2b, can be obtained according to the following integration (7, 9).

$$\Delta\lambda_{ij} = \frac{2(1-\nu^2)}{E}\int_0^A \left[\frac{K_{Ij}}{F_j}\frac{\partial K_{Ii}}{\partial F_i} + \frac{K_{IIj}}{F_j}\frac{\partial K_{IIi}}{\partial F_i} + \frac{K_{IIIj}}{F_j}\frac{\partial K_{IIIi}}{\partial F_i}\right] dA$$

Where $\Delta\lambda_{ij}$ is the displacement in direction i due to unit force applied in direction j (flexibility or compliance), K_{Ij}, K_{IIj} and K_{IIIj} are three modes stress intensity factors due to load applied in direction j. K_{Ii}, K_{IIi} and K_{IIIi} are the three modes stress intensity factors due to load F_i applied in the direction of calculated deformation. And A is the crack surface area.

Increases in compliance (flexibility) in directions of various degrees of freedom given in Fig. 2b can now be obtained by substituting from equations 1-4 into the above integration as follows:

$$\Delta\lambda_{pp} = \frac{2(1-\nu^2)}{E}\int_o^A \left[\frac{K_{IP}}{P}\frac{\partial K_{IP}}{\partial P}\right] dA$$

$$\Delta\lambda_{pp} = \frac{2(1-\nu^2)}{EB}\,[1.98n^2 - 0.544n^3 + 18.65n^4 - 33.697n^5 + 99.26n^6 - 211.9n^7 + 436.84n^8 - 460.48n^9 + 289.98n^{10}] \quad (5)$$

$$\Delta\lambda_{mm} = \frac{2(1-\nu^2)}{E}\int_0^A \left[\frac{K_{IM}}{M}\cdot\frac{\partial K_{IM}}{\partial M}\right] dA$$

$$\Delta\lambda_{mm} = \frac{72(1-\nu^2)}{EBd^2}\,[1.98n^2 - 3.27n^3 + 14.43n^4 - 31.26n^5 + 63.56n^6 - 103.36n^7 + 147.52n^8 - 127.69n^9 + 61.5n^{10}] \quad (6)$$

$$\Delta\lambda_{mp} = \frac{2(1-\nu^2)}{E}\int_0^A \left[\frac{K_{IP}}{P}\frac{\partial K_{IM}}{\partial M}\right] dA = \Delta\lambda_{pm}$$

$$\Delta\lambda_{mp} = \frac{12(1-\nu^2)}{EBd}\,[1.98n^2 - 1.91n^3 + 16n^4 - 34.84n^5 + 83.93n^6 - 153.65n^7 \; 256.72n^8 - 244.67n^9 + 133.55n^{10}] \quad (7)$$

$$\Delta\lambda_{vv} = \frac{2(1-\nu^2)}{E}\int_0^A \left[\frac{K_{IIV}}{V}\frac{\partial K_{IIV}}{\partial V}\right] dA$$

$$\Delta\lambda_{vv} = \frac{2(1-\nu^2)}{EB}\,[1.98n^2 + 1.83n^5 + 0.66n^8] \quad (8)$$

$$\Delta\lambda_{tt} = \frac{2(1-\nu^2)}{E}\int_0^A \left[\frac{K_{IIIt}}{M_t}\;\frac{\partial K_{IIIt}}{\partial M_t}\right] dA$$

$$\Delta\lambda_{tt} = \frac{22.2(1-\nu^2)}{EB^3}\,[1.57n^2 - 2.67n^3 + 1.27n^4] \quad (9)$$

It should be noticed that the remaining compliances (flexibilities); $\Delta\lambda_{pv}$, $\Delta\lambda_{mv}$, $\Delta\lambda_{pt}$, $\Delta\lambda_{mt}$ and their reciprocal terms are all equal zero due to the fact that K_{IV}, K_{It}, etc. are equal zero.

Flexibility and Stiffness Matrices of Cracked Section

Based on the crack surface displacements relations obtained above, relations between forces and displacements in the form of a flexibility matrix can be simulated. Four degrees of freedom are assumed at both ends of the cracked segments as shown in Fig. 2b. These degrees of freedom correspond to axial (13, 19), shear (14, 20), torsion (16, 21) and bending (18, 22) deformations. The flexibility matrix corresponding to the left end of the cracked segment, I, can then be assembled as follows:

$$[F_{II}] = \begin{array}{c} \\ 13 \\ 18 \\ 16 \\ 14 \end{array} \begin{array}{c} \begin{array}{cccc} 13 & 18 & 16 & 14 \end{array} \\ \begin{bmatrix} \Delta\lambda_{pp} & \Delta\lambda_{pm} & 0 & 0 \\ \Delta\lambda_{mp} & \Delta\lambda_{mm} & 0 & 0 \\ 0 & 0 & \Delta\lambda_{tt} & 0 \\ 0 & 0 & 0 & \Delta\lambda_{vv} \end{bmatrix} \end{array}$$

The inverse of the above flexibility matrix is the stiffness matrix corresponding to the end I, which is given as follows:

$$[S_{II}] = \begin{array}{c} \\ 13 \\ 18 \\ 16 \\ 14 \end{array} \begin{array}{c} \begin{array}{cccc} 13 & 18 & 16 & 14 \end{array} \\ \left[\begin{array}{cccc} S_{pp} & S_{pm} & 0 & 0 \\ S_{mp} & S_{mm} & 0 & 0 \\ 0 & 0 & S_{tt} & 0 \\ 0 & 0 & 0 & S_{vv} \end{array}\right] \end{array}$$

where,

$$S_{pp} = \Delta\lambda_{mm}/D, \quad S_{mm} = \Delta\lambda_{pp}/D$$
$$S_{mp} = S_{pm} = -\Delta\lambda_{pm}/D, \quad S_{tt} = 1/\Delta\lambda_{tt},$$
$$S_{vv} = 1/\Delta\lambda_{vv}, \quad D = \Delta\lambda_{pp} \cdot \Delta\lambda_{mm} - \Delta\lambda_{mp}^{2}$$

Then the stiffness matrix of the element 2 is shown in Fig. 1b representing the cracked section $(K)_c$ can be written in the form:

$$[K]_c = \begin{array}{c} \begin{array}{cccccccc} 13 & 18 & 16 & 14 & 19 & 22 & 21 & 20 \end{array} \\ \left[\begin{array}{cccccccc} S_{pp} & S_{pm} & 0 & 0 & -S_{pp} & -S_{pm} & 0 & 0 \\ S_{mp} & S_{mm} & 0 & 0 & -S_{mp} & -S_{mm} & 0 & 0 \\ 0 & 0 & S_{tt} & 0 & 0 & 0 & -S_{tt} & 0 \\ 0 & 0 & 0 & S_{vv} & 0 & 0 & 0 & -S_{vv} \\ -S_{pp} & -S_{pm} & 0 & 0 & S_{pp} & S_{pm} & 0 & 0 \\ -S_{mp} & -S_{mm} & 0 & 0 & S_{mp} & S_{mm} & 0 & 0 \\ 0 & 0 & -S_{tt} & 0 & 0 & 0 & S_{tt} & 0 \\ 0 & 0 & 0 & -S_{vv} & 0 & 0 & 0 & S_{vv} \end{array}\right] \end{array} \begin{array}{c} \\ 13 \\ 18 \\ 16 \\ 14 \\ 19 \\ 22 \\ 21 \\ 20 \end{array} \qquad (10)$$

STIFFNESS MATRIX OF CRACKED MEMBER

Three stiffness matrices are mainly used to assemble the overall stiffness matrix of size 22 × 22 for the cracked member shown in Fig. 1. The first and third stiffness matrices are of size 12 × 12, corresponding to the first and third common space frame elements of lengths L_1 and L_2 respectively. The second stiffness matrix simulates the cracked segment is given by equation 10 and of size 8 × 8 corresponding to the degrees of freedom marked in equation 10. These three matrices are all located in the overall stiffness matrix in the corresponding positions referred to the chosen system of numbering for the degrees of freedom. Thus, the final form of the cracked member matrix (K) can be given in the following form:

$$[K]_{22x22} = \begin{array}{c} \begin{array}{cc} 1\ldots12 & 13\ldots22 \end{array} \\ \left[\begin{array}{c|c} A_{12x12} & B_{10x10} \\ \hline B^{T}_{10x10} & C_{10x10} \end{array}\right] \end{array} \begin{array}{c} 1 \\ \vdots \\ 12 \\ 13 \\ \vdots \\ 22 \end{array} \qquad (11)$$

Where [A], [B] and [C] are matrices given in the Appendix. The above matrix can be condensed to a size of 12 × 12 by assuming the two intermediate nodes are unloaded. Hence, the stiffness matrix (K_f) relating forces and displacements at the exterior nodes of the cracked member shown in Fig. 1 can be written as follows:

$$[K_f]_{12\times12} = [A] - [B]\,[C]^{-1}\,[B]^T \quad (12)$$

Thus the matrix given by equation 12 is similar in dimensions to the stiffness matrix of the common space frame member, and this matrix can be employed directly in the analysis of space structures containing cracked members using the stiffness method. This matrix can be assembled in the overall stiffness matrix of the whole structure together with the stiffness matrix of the common uncracked space frame members in the same manner.

EXAMPLES

Based on the method of analysis developed above, a computer program has been written to analyze three dimensional skeletal structures containing cracked and uncracked members. It should be noted that the present computer program is valid for any number of cracked sections provided that cracks are assumed at the tension side of the element cross section. This program can also be employed in the special cases of plane frames and plane grids by choosing the appropriate coordinate system of axes as shown below.

The present model has been employed to analyze three examples; plane frame, plane grid and space frame as shown in Fig. 4-7. Geometry, dimensions, loadings, and locations of cracks chosen in these problems are indicated in these figures.

Fig. 5 shows the variations of bending moment M_1/M_{10} at the upper left cracked corner, variation of bending moment M_2/M_{20} at the upper right corner and the variation of horizontal deflection δ_1/δ_{10} with crack depth ratio a/d in plane frame problem. It should be noted that M_{10} and M_{20} are the original bending moment values at the upper left and right corners respectively for crack depth equal zero. Also δ_0 is the original horizontal deflection at frame corners when crack size is equal zero.

As the crack depth ratio increases, the bending moment, M_1 decreases, bending moment, M_2 increases and deflection, δ increases as shown in Fig. 5. This redistribution in internal forces and increase in deflection occurred due to the presence of the crack and is more pronounced as the crack depth increase.

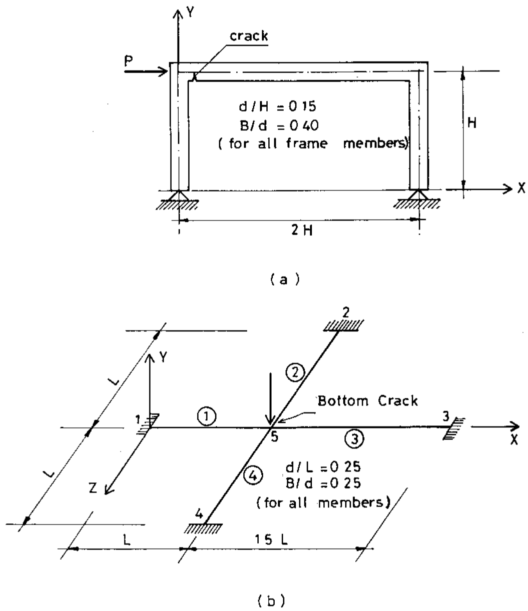


Fig. 4: Examples of Plane Structures. (a) Plane frame example. (b) Plane grid example.

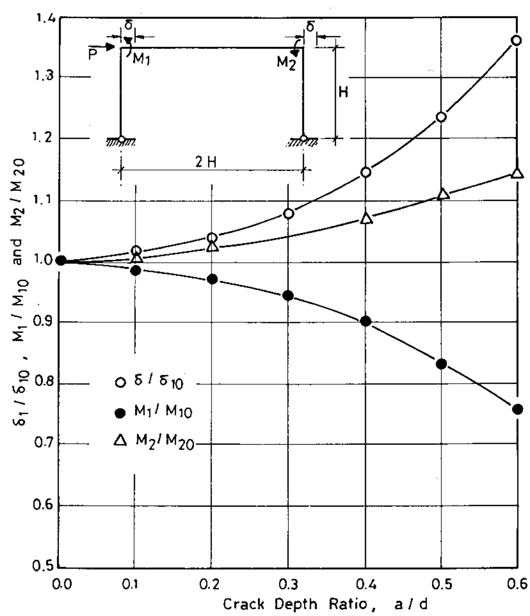


Fig. 5: Variation of bending moments and deflection due to crack presence in plane frame.

Fig. 6 gives the variations in bending moments and torsional moments at various sections of the shown plane grid cracked at the end of member 2 as shown in Fig. 4. These variations are presented using a non-dimensional parameters by dividing the instantaneous values by the original values corresponding to zero crack size. It is noticed that while bending moment and torsional moments decreased at cracked section, it increased at other locations as shown in the figure and the effect is more pronounced as the crack depth ratio increased.

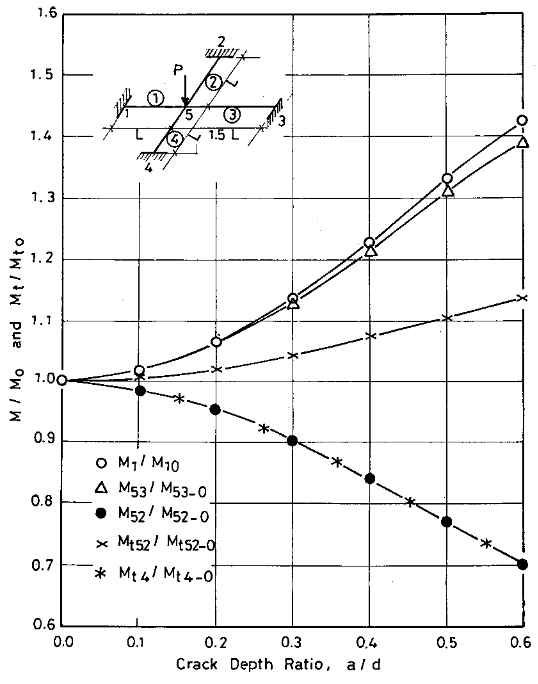


Fig. 6: Variation of bending and torsional moments due to crack presence in plane grid.

Fig. 7 presents the effect of crack presence at the end of member 8 in the space frame example, on the internal bending and torsional moments at various frame joints. It is noticed that while a decrease in bending and torsional moments has occurred at the cracked section, an increase in these moments has occurred at other uncracked sections.

Comparing the results given in Fig. 7 with those corresponding to the plane problems, discussed above, it is noticed that the variation in internal forces in the cases of plane problems is much larger than in the case of space frame problems due to crack presence. In addition, the variations in moment values only occurred locally at points around the cracked section of the space frame.

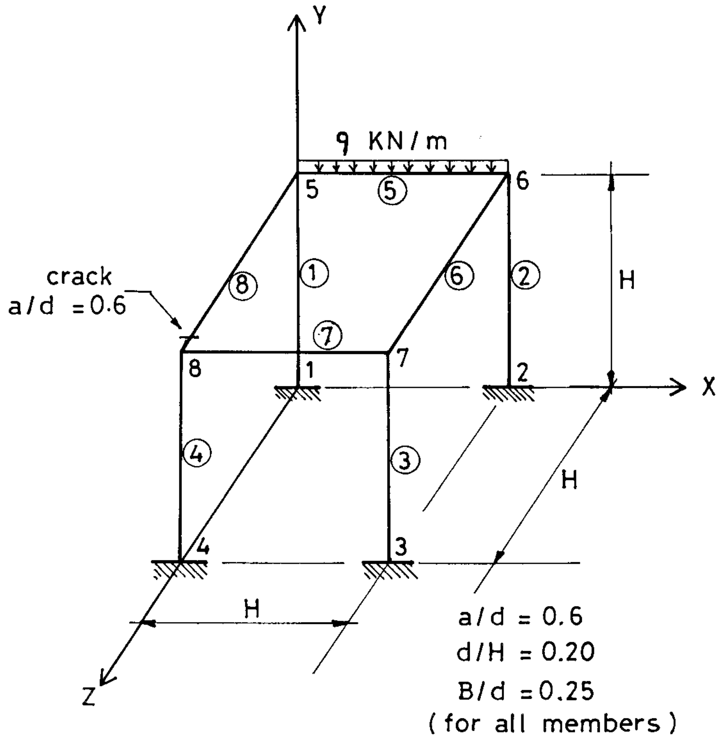


Fig. 7: Space frame example.

Joint	Member	M/M_0	M_t/M_{t0}
8	8	0.92	0.89
6	2	1.01	1.003
7	3	1.02	1.05
6	6	1.05	1.10

CONCLUSIONS

Based on the stiffness matrix method and fracture mechanic techniques, a computer model has been developed to analyze three dimensional skeletal structures containing cracked members. This model takes into consideration crack surface displacements produced in mixed modes of cracking due to presence of axial force, bending moment, shearing force and torsional moments. This model can also be applied in the special cases of plane frames and plane grids.

Based on the results of three examples of plane frame, plane grid and space frame cracked at various locations, several conclusions can be made:

1. A redistribution of bending moments and torsional moments and deflections occurs in plane and space structural members due to the presence of crack.
2. The effect of cracking on the variation of internal forces and deformations is more pronounced in plane type of structures compared with space structures where the variation is restricted to points around the cracked section.
3. Bending and torsional moments are reduced at cracked sections and increased at other locations of uncracked sections depending on crack location, crack depth and geometry of the structure.
4. The deformations in studied structures are shown to increase as the crack size increases.
5. The present analysis is of use in identifying overstressed zones in buildings and structures which must be strengthened due to cracking.

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APPENDIX: STIFFNESS MATRIX OF CRACKED SPACE MEMBER

The stiffness matrices given in equations 11, 12 are given below:

Matrix [A]

[A] = A(I,J)

I = 1,12 and J = 1,12

All elements in matrix [A] are equal zero except the following:

$A(1,1) = EA_x/L_1$

$A(2,2) = 12EI_z/L_1^3$

$A(3,3) = 12EI_y/L_1^3$

$A(4,4) = GJ/L_1$

$A(5,5) = 4EI_y/L_1$

$A(6,6) = 4EI_z/L_1$

$A(7,7) = EA_x/L_2$

$A(8,8) = 12EI_z/L_2^3$

$A(9,9) = 12EI_y/L_2^3$

$A(10,10) = GJ/L_2$

$A(11,11) = 4EI_y/L_2$

$A(12,12) = 4EI_z/L_2$

$A(2,6) = A(6,2) = 6EI_z/L_1^2$

$A(3,5) = A(5,3) = -6EI_y/L_1^2$

$A(8,12) = A(12,8) = -6EI_z/L_2^2$

$A(9,11) = A(11,9) = 6EI_y/L_2^2$