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STABILITY OF LOGISTIC AUTOREGRESSIVE MODEL

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استقرارية نموذج الانحدار الذاتي اللوجستي

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في هذه الورقة اقترحنا نموذج الانحدار الذاتي اللوجستي والذي هو حالة خاصة من نماذج الانحدار الـــذاتي متعـــدد الحدود . الهدف من هذه الورقة هو دراسة استقرارية هذا النموذج وإيجاد شروط استقراريته باستخدام أسلوب ديـــناميكي (حركي) منسوب للعالم اوزاكي <u>Ozaki</u> وتطبيق هذه الشروط على البيانات الخاصة بعدد الحالات الشهرية لمرض حمـــى مالطا المسجلة في العراق للفترة (1989-2002).

Key words: *Limit cycle , Logistic Autoregressive Model Singular point, Stability*

ABSTRACT

In this paper we propose a Logistic Autoregressive Model, which is a special case of polynomial Autoregressive Model. The aim of this paper is to study and find the stability conditions of the above model by using a dynamical approach due to <u>Ozaki</u> and apply these conditions to the monthly-recorded Brucellosis data in Iraq in the interval (1989-2002).

Introduction

In the last three decades there was a growing interest in the study of nonlinear time series models. The first and simplest reason is that the world is mostly nonlinear.

Here we are interested in nonlinear dynamics, that is, in the nonlinear relationships existing between observations of an object made sequentially over the time, where there is a theoretical evidence against linearity or jump phenomena.

The general representation of a nonlinear time series of order p and q is given by:

$$x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-p}, \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-q}, \varepsilon_t), \qquad t \geq 1 \ldots \qquad (1)$$

where $\{\varepsilon_t\}_{t \in N}$ be a sequence of independent and identically distributed (IID) random variables and $f : R^{p+q+1} \rightarrow R$ is a nonlinear function.The Nonlinear Autoregressive (NLAR) models are the most famed models among the nonlinear time series models. The general form of a nonlinear autoregressive model of order p represents as

$$x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-p}) + \varepsilon_t$$

where $\varepsilon_t \sim IID$ white noise process, and $f : R^p \rightarrow R$ is a nonlinear function.

The study of stability, *i.e.* stationarity of nonlinear autoregressive models leads often to stronger sufficient conditions on the coefficients than in the linear models, because zero is the only fixed point in linear AR models. While, in NLAR models there are many fixed points of each model.

The literature discussing the stationarity and ergodicity of the NLAR (p) can be roughly divided into two categories: general cases and special cases. For the general cases Chan and Tong (1985), Tong (1990), An and Huang (1996) and Fonseca (2000) have developed many good tools for finding a general stability conditions of nonlinear AR(p) models. Since they are dealing with general cases, the conditions given in these articles are usually very general (See [1], [2], [3], [4]).

For the special cases, Chan and Tong (1985) and Tong (1990) give the sufficient and necessary conditions for the geometric ergodicity of the threshold autoregressive model. Ozaki (1982) presents a sufficient condition for the erogodicity of exponential AR models. To insure the geometrical ergodicity of the NLAR model they present a number of conditions based on a Lipschitz continuity of a nonlinear function.

Ozaki (1982) proposes a dynamical approach in order to find the necessary and sufficient conditions for stability of exponential autoregressive models. This approach is a local linearization technique used to find the approximated linear autoregressive model near the nonzero singular point of a nonlinear model. By using a variational equation near the nonzero singular point of an exponential autoregressive model, Ozaki finds the sufficient and necessary condition for existence and stability of a non-zero singular point of EXPAR model. Furthermore, he finds the stability conditions of a limit cycle if it exists (See [5], [6]).

The stability conditions of nonlinear time series model consists of the stationarity conditions of the model at the zero singular point and the stability conditions of a non-zero singular point if it exists. Otherwise, the stability conditions of limit cycle when the model posses a limit cycle.To study the stability conditions in terms of EXPAR(p) model parameters Ozaki proposed a dynamical approach based on a local linearization approximation of the model in the neighborhood of a non zero fixed point of the model. In order to discuss this approach we need the following two definitions for general discrete time difference equation. $x_t = f(x_{t-1}, x_{t-2}, \ldots\ldots, x_{t-p}) \quad \ldots \qquad (2)$

Definition 1: [5] and [6]

A singular point ξ of model (2) is defined as a point, for which every trajectory of the model (2) beginning sufficiently near ξ approaches it either for $t \rightarrow \infty$ or $t \rightarrow -\infty$. If it approaches ξ for $t \to \infty$ we call ξ a stable singular point, and if it approaches ξ for $t \to -\infty$ we call ξ an unstable singular point.

Definition 2: [5] and [6]

A limit cycle of model (2) is defined as an isolated and closed trajectory $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q} = x_t$, where q is a positive integer. Closed means that if the initial value $(x_1, x_2, \ldots\ldots, x_p)$ belongs to the limit cycle, then $(x_{1+kq}, x_{2+kq}, \ldots\ldots, x_{p+kq}) = (x_1, x_2, \ldots\ldots, x_p)$ for any integer k. Isolated means that every trajectory beginning sufficiently near the limit cycle approaches it either for $t \to \infty$ or $t \to -\infty$. If it approaches the limit cycle for $t \to \infty$ we call it a stable limit cycle, and if it approaches the limit cycle for $t \to -\infty$ we call it an unstable limit cycle. The smallest integer q that satisfies definition (2) is called the period of the limit cycle.

Logistic Autoregressive Model

The logistic map plays a role in nonlinear dynamical systems in general, and in chaotic dynamical systems as a famed map, because this map possess an important dynamical properties related to the behavior of trajectories generated by this map.

The logistic map has the form,

$$x_t = ax_{t-1}(1 - x_{t-1}) \qquad x_t \in [0,1] \quad \ldots \qquad (3)$$

Clearly for $0 \le a \le 4$,equation (3) defines a map from the unit interval into itself. [7], [4]

Tong in 1990 discusses threshold autoregressive models such as,

$$x_t = \alpha(x_{t-k})x_{t-1} + \varepsilon_t \ \ldots \qquad (4)$$

where

$$\alpha(x_{t-k}) = \begin{cases} \alpha_1 & if \quad x_{t-k} > a \\ \alpha_2 & if \quad x_{t-k} \le a \end{cases}$$

where ε_t is a white noise input. [8], [4]

Note that for $0 \le a \le 4$ in (3) the trajectories lies inside the unit interval, which mean that $x_t \in [0,1]$ for any $t = 0,1,\ldots\ldots$ then we define a logistic model such as

$$x_t = ax_{t-1}(1 - x_{t-1}) + \varepsilon_t, \quad x_t \in [0,1], 0 \le a \le 4 \ \ldots \qquad (5)$$

where ε_t is a white noise input.

In fact, model (5) is a special case of polynomial autoregressive models of order 1 with restriction that $0 \le a \le 4$.

Clearly zero is a singular point of model (5). For a first order nonlinear autoregressive model (p=1), Fonseca (2000) mentions that the Markov chain related to a nonlinear autoregressive model is geometrically ergodic if there exists r>0 such that

$$\sup_{|x|>r} \left| \frac{f(x)}{x} \right| < 1 \quad \ldots \qquad (6)$$

where $x_t = f(x_{t-1})$ [3].

In our model

$$\left|\frac{f(x_{t-1})}{x_{t-1}}\right| = \left|\frac{ax_{t-1}(1-x_{t-1})}{x_{t-1}}\right| \le a\left|(1-x_{t-1})\right|$$

and for any real constant $0 < r < 1$

$$\sup_{|x_{t-1}|>r}\left|\frac{f(x_{t-1})}{x_{t-1}}\right| \le a \quad \ldots \quad (7)$$

Because $|1-x_{t-1}| \le 1$,then the Markov chain associated to model (5) is geometrically ergodic if $a < 1$. This is a concordant result to those mentioned in [9] related to the logistic map.

We can easily find that the non-zero singular point of model (5) is given by

$$\xi = 1 - \frac{1}{a} \quad \ldots \quad (8)$$

We can find the stability condition of model (5) at this non-zero singular point by using a variational difference equation, *i.e.*

by replacing $\xi + \xi_t, \xi + \xi_{t-1}$ instead of x_t and x_{t-1} and suppressing the white noise in model (5). For $|\xi_t|, |\xi_{t-1}|$ sufficiently small, we get

$$\xi_t = (2-a)\xi_{t-1} \quad \ldots \quad (9)$$

Then the stability condition of the model at the nonzero singular point ξ is given by $|2-a| < 1$ which yield that $1 < a < 3$ which is the same condition mentioned in [9].

Using the same way, we can find the stability condition of limit cycle if it exists in model (5).

Let the limit cycle of period q of the model (5) has the form $x_t, x_{t+1}, x_{t+2}, \ldots\ldots, x_{t+q} = x_t$

Points x_t, x_{t-1} on a trajectory near the limit cycle are represented as $x_t + \xi_t, x_{t-1} + \xi_{t-1}$ for $|\xi_t|, |\xi_{t-1}|$ sufficiently small respectively, we find the following difference equation

$$\xi_t = a(1-2x_{t-1})\xi_{t-1} \quad (10)$$

Now (10) represents a difference equation with periodic coefficients, which is difficult to solve analytically. What is required is to know whether ξ_t of (10) converges to zero or not ,and this can be checked by seeing whether $|\xi_{t+q}/\xi_t|$ less than one or not. From the relation (10) we get

$$\xi_{t+q} = [a(1-2x_{t+q-1})]\xi_{t+q-1}$$

$$= [a(1-2x_{t+q-1})].[a(1-2x_{t+q-2})]\ldots\ldots \quad [a(1-2x_t)]\xi_t$$

$$= \left[\prod_{j=1}^{q} a(1-2x_{t+q-j})\right]\xi_t$$

Implies that

$$\frac{\xi_{t+q}}{\xi_t} = a^q.\left[\prod_{j=1}^{q}(1-2x_{t+q-j})\right] \ldots \quad (11)$$

This leads us to construct the following proposition.

Proposition 1:

Provided its existence, a limit cycle of period q the logistic model (5) is orbitally stable if

$$\left|\prod_{j=1}^{q}(1-2x_{t+q-j})\right| < \left(\frac{1}{a}\right)^{q} \quad \dots \qquad (12)$$

The result of this proposition is very important to test the stability of a limit cycles of the logistic model if it exists by applying the condition in equation (12), here a parameter a and the q values of a limit cycle $x_t, x_{t+1}, \dots\dots\dots, x_{t+q-1}$ play a major rule to decide the stability of the model.

Application And Examples

Brucellosis is well-known disease in hot areas around the world such as Africa, South America, Middle East and Mediterranean Sea zone. In Iraq the number of Brucellosis cases increases between May and September in each year. This fact is clearly appeared in figure (1), the time series plot of monthly observations of Brucellosis cases in Iraq (1989-2002) [10].

Figure (1) shows a strongly seasonal component with large values in summer seasons. From the above description of data, we can say that the non-linearity clearly appeared with its characteristic phenomena

In our applied work, we use STATISTICA software (nonlinear estimation and time series parts) in estimating the model parameters and modeling the linear autoregressive model and we construct a C_{++} program to plot the behavior of trajectories (simulation plot) for a large number of iterations.

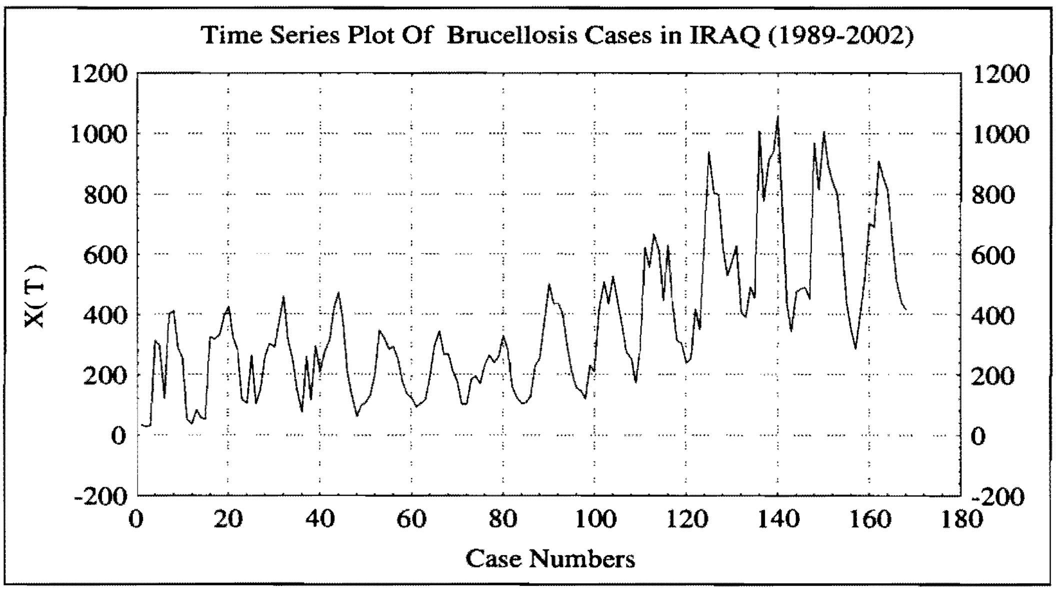


Figure 1: Time series plot of Brucellosis Data in Iraq (1989-2002)

For logistic modeling, all data are normalized to $[0,1]$ interval by using the following equation:

$$x_i = \frac{i\text{th observatio n}}{\text{maximum observatio n}}, \qquad i = 1,\ 2,\ \ldots,\ 168 \qquad (13)$$

Figure (2) shows the time series plot of transformed data according to equation (13). After we estimate the parameter of logistic model, we obtain the following model.

$$x_t = 1.880137 x_{t-1}(1 - x_{t-1}) + \varepsilon_t,\ x_t \in [0,1] \ \ldots \qquad (14)$$

$$\hat{\sigma}_\varepsilon^2 = 0.048985$$

where $\hat{\sigma}_\varepsilon^2$ denotes the residuals variance of the model.

The dimensional plot of the model (14) and the normal probability plot of residuals appear in figure (3).

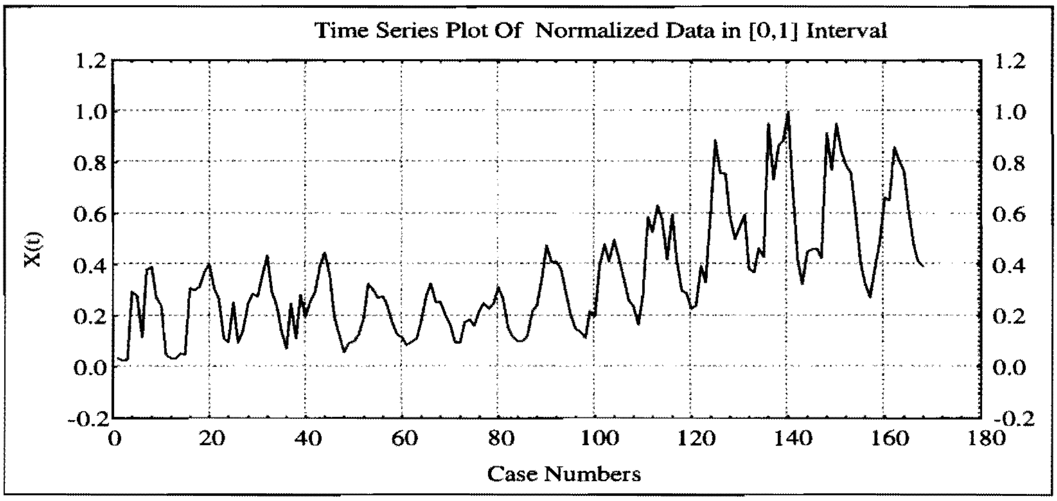
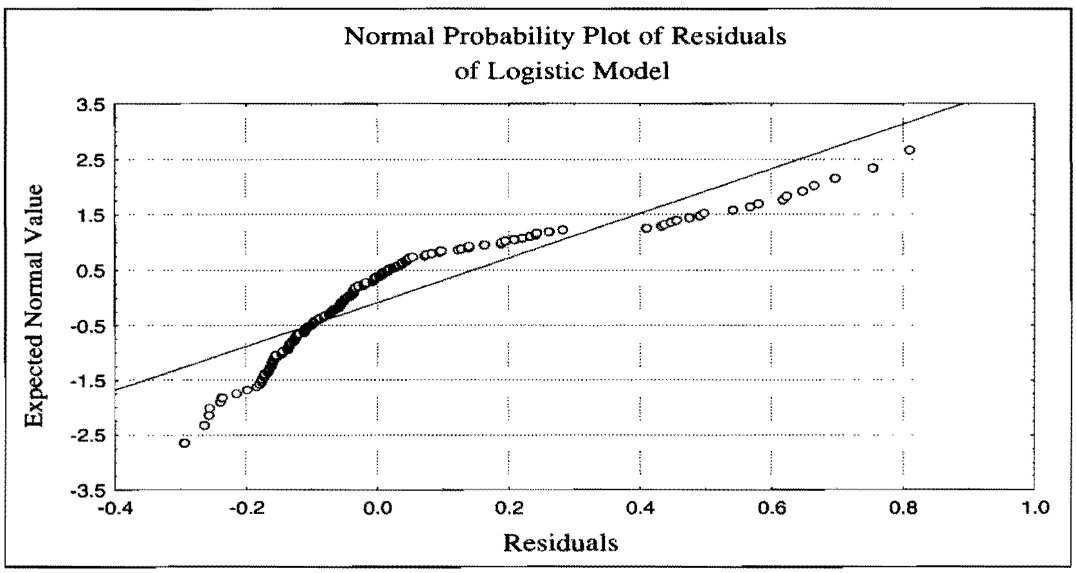


Figure 2: Time series plot of normalized data in [0,1] interval



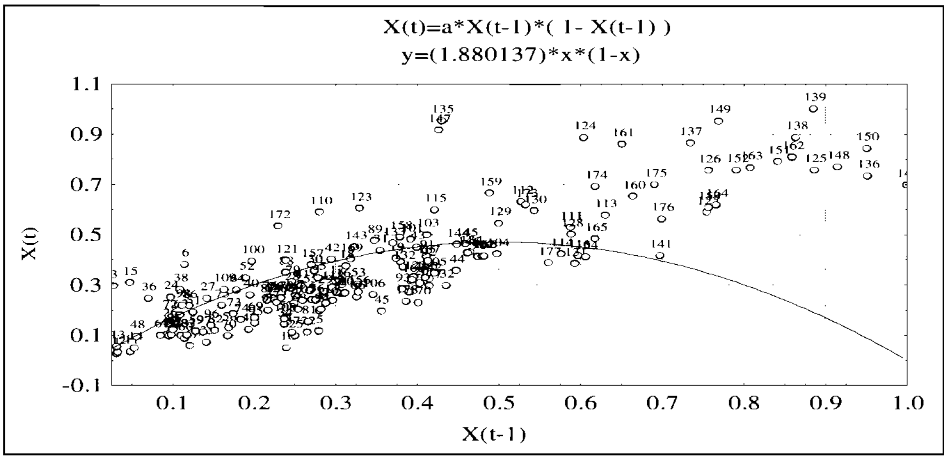


Figure 3: Logistic Autoregressive Model (14) and Normal Probability Plot of Residuals

By comparing model (14) with stability condition ($1<a<3$) , we say that the model (14) have stable nonzero singular point ($\xi = 0.4661$).

This result is checked by the simulation plot shown in figure (4), different trajectories start from the different initial values x_0 approach the nonzero singular point as $t \to \infty$.

In addition, we calculate the Akaike's information criterion for nonlinear autoregressive models given by

$$AIC = (N-m)\ln(\hat{\sigma}_{\varepsilon}^{2}) + 2(number \quad of \quad parameters) \quad \ldots \qquad (15)$$

where N is the number of observations, $\hat{\sigma}_{\varepsilon}^{2}$ is the residuals variance of the model and m is the maximum lag in all models in the set of models being considered. [11]

For model (14) we obtain that.

AIC (1) = -502.712

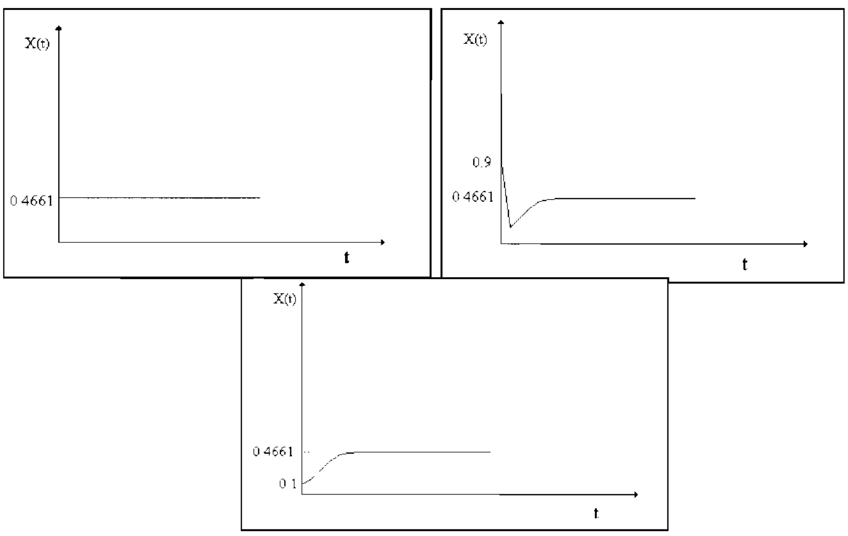


Figure 4: Time series plots of simulated realizations of model (14) Starting with difference initial values x_0

To make a comparison between the autoregressive logistic model and the other nonlinear autoregressive model of the first order see [12]. We choose the exponential autoregressive model of the first order EXPAR (1) for modeling the normalized data in [0, 1] interval and we find the following model

$$x_t = (0.204062 + 0.851628 e^{-0.279354 x_{t-1}^2}) x_{t-1} + \varepsilon_t \quad (16)$$

where ε_t is a white noise and $\hat{\sigma}_\varepsilon^2$ is the residuals variance of the model . Figure (5) shows the two dimensional plot and Normal Probability Plot of Residuals of EXPAR (1) model (16).

We calculate the Akaike's information criterion for exponential autoregressive model by using equation (15) and we find that

AIC (1) =-351.139

According to the values of $\hat{\sigma}_\varepsilon^2$ and AIC (1) of the logistic autoregressive model and EXPAR(1) model , the logistic autoregressive model is the best fitted among the two models

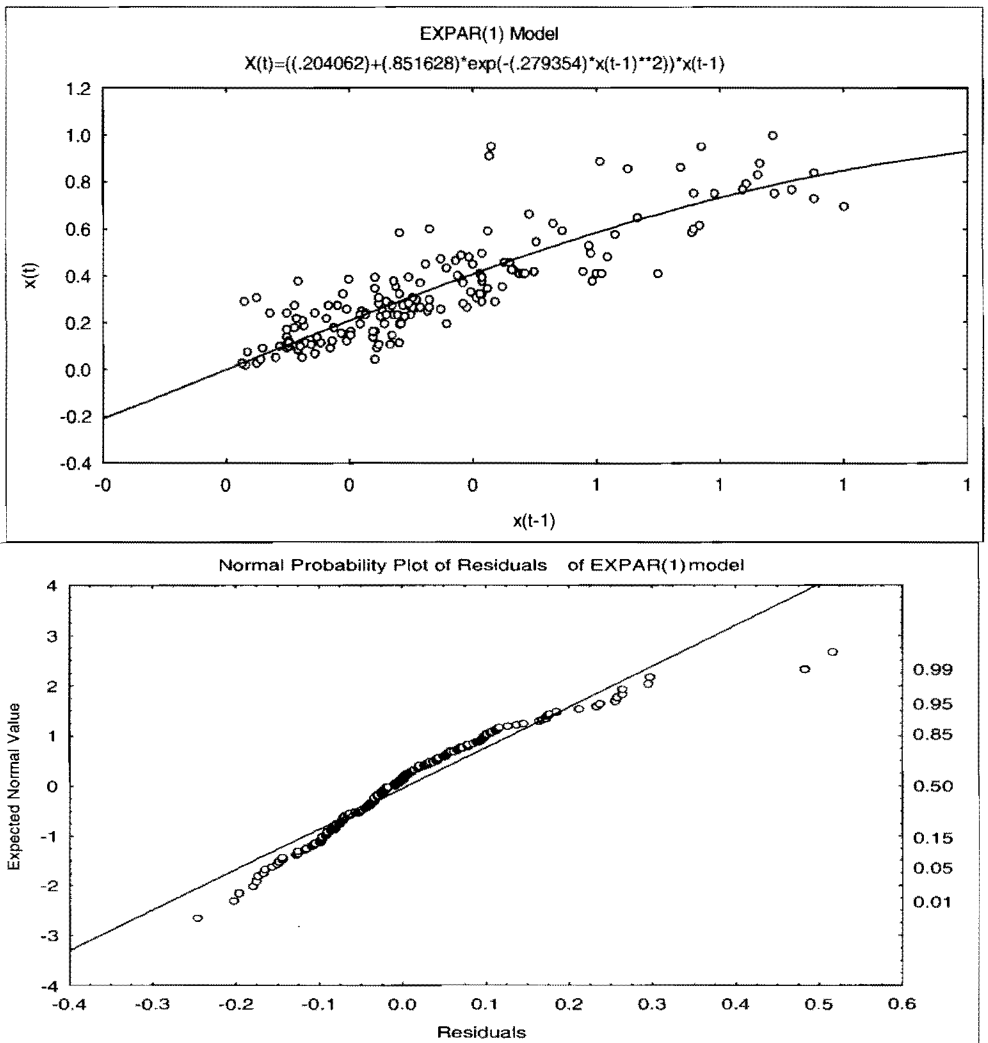


Figure 5: two dimensional plot and Normal Probability Plot of Residuals of EXPAR (1) model (16).

Since we are not faced any limit cycle in our application to due to the Brucellosis data, We apply this result of proposition (1) to the following examples for an arbitrary values of a parameter a and we check the stability of logistic models mentioned in the following two examples.

Example 1:

The logistic model given by

$$x_t = 3.8x_{t-1}(1-x_{t-1})+\varepsilon_t, x_t \in [0,1] \ldots \qquad (17)$$

has unstable non-zero singular point, $\xi = 0.7435$,and a limit cycle of period 5 which is

$\{0.93, 0.55, 0.82, 0.68, 0.23, 0.93\}\ldots$ (18)

by easy calculations we see that

$$\sum_{i=1}^{5}\left(1-2x_{t+5-j}\right)=0.01107, \qquad \left(\frac{1}{3.8}\right)^{5}=0.1262X10^{-2}$$

Then the condition (12) does not satisfy, and the limit cycle (18) is orbitally unstable.

Figure (6) shows that the trajectories starting from different initial values tends to unstable limit cycle as $t \to \infty$.

Example 2:

The logistic model given by

$$x_t = 3.2x_{t-1}(1-x_{t-1})+\varepsilon_t \qquad x_t \in [0,1] \ldots \qquad (19)$$

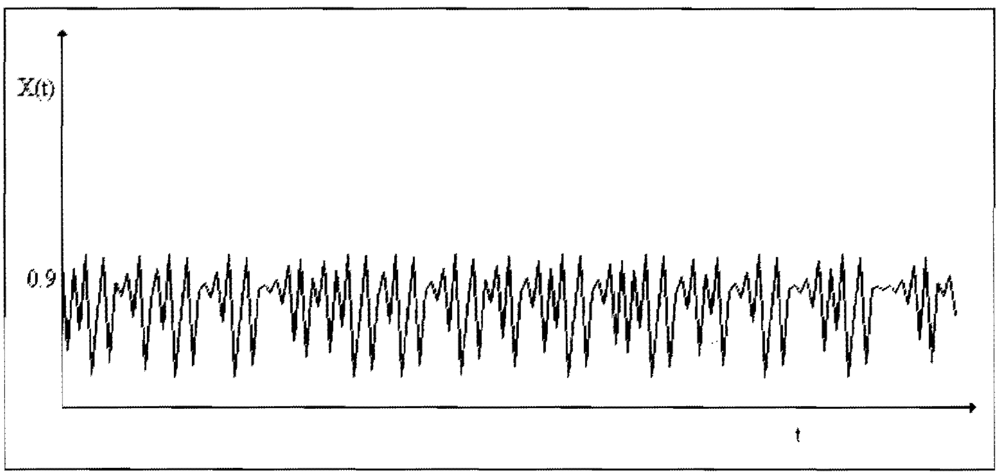
has an unstable non-zero singular point $\xi = 0.6875$

The trajectories starting from different initial value x_0 tend to a limit cycle of period 2, the trajectory oscillate between the two values 0.513 and 0.7994 by easy calculations we see that

$$\left(\frac{1}{a}\right)^{2} = 0.0976 \text{ and}$$

$$(1-2\times 0.513).(1-2\times 0.7994)=0.01557$$

Then the condition (12) is satisfied, and by proposition (1), we obtain that the limit cycle of model (19) is orbitally stable. Figure (7) shows that the trajectories starting from different initial value x_0 approaches the limit cycle as $t \to \infty$.



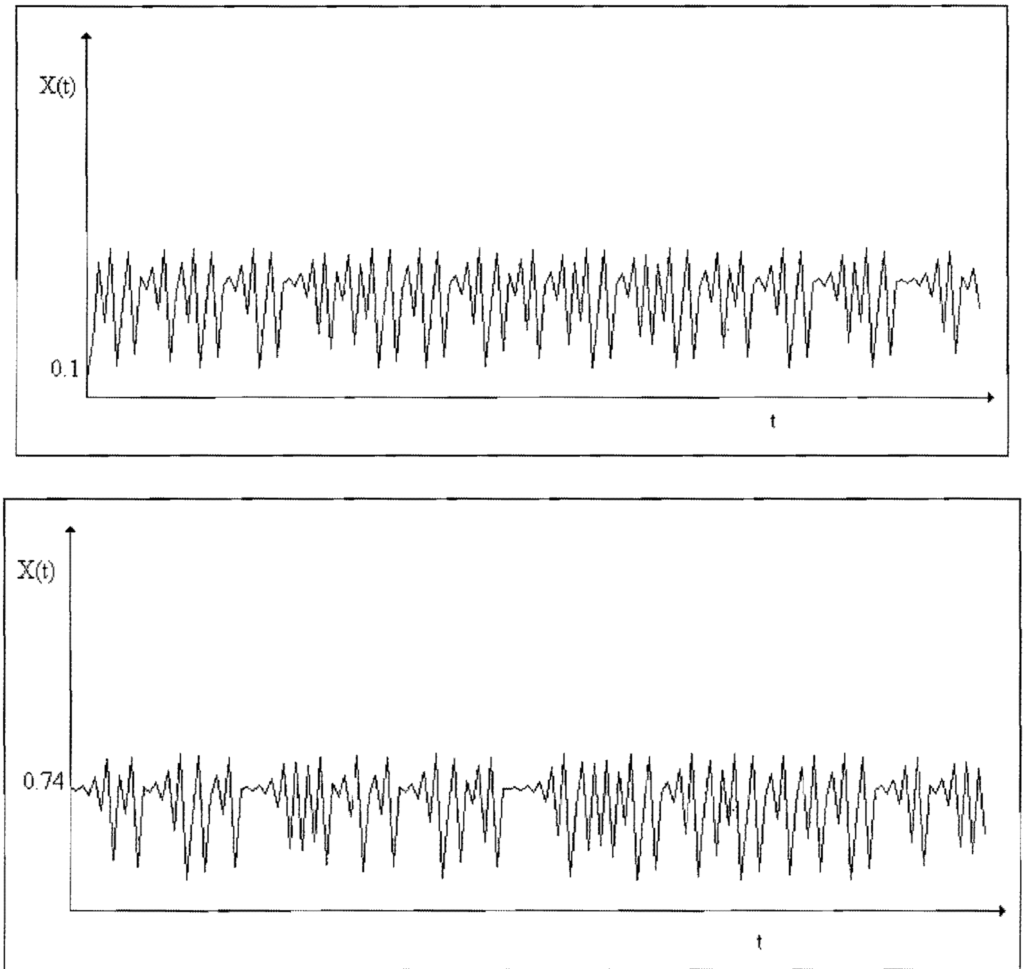
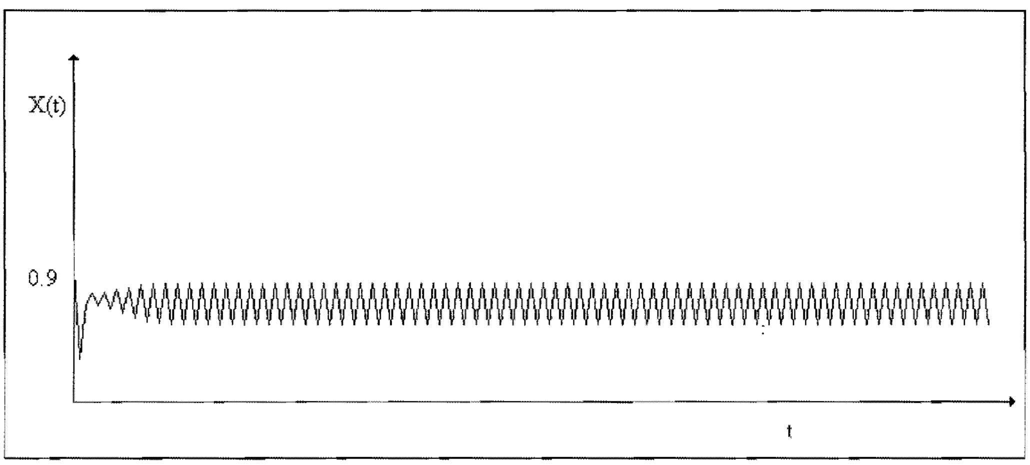


Figure 6: Time series plots of simulated realizations of the model in example (1) Starting with difference initial values x_0



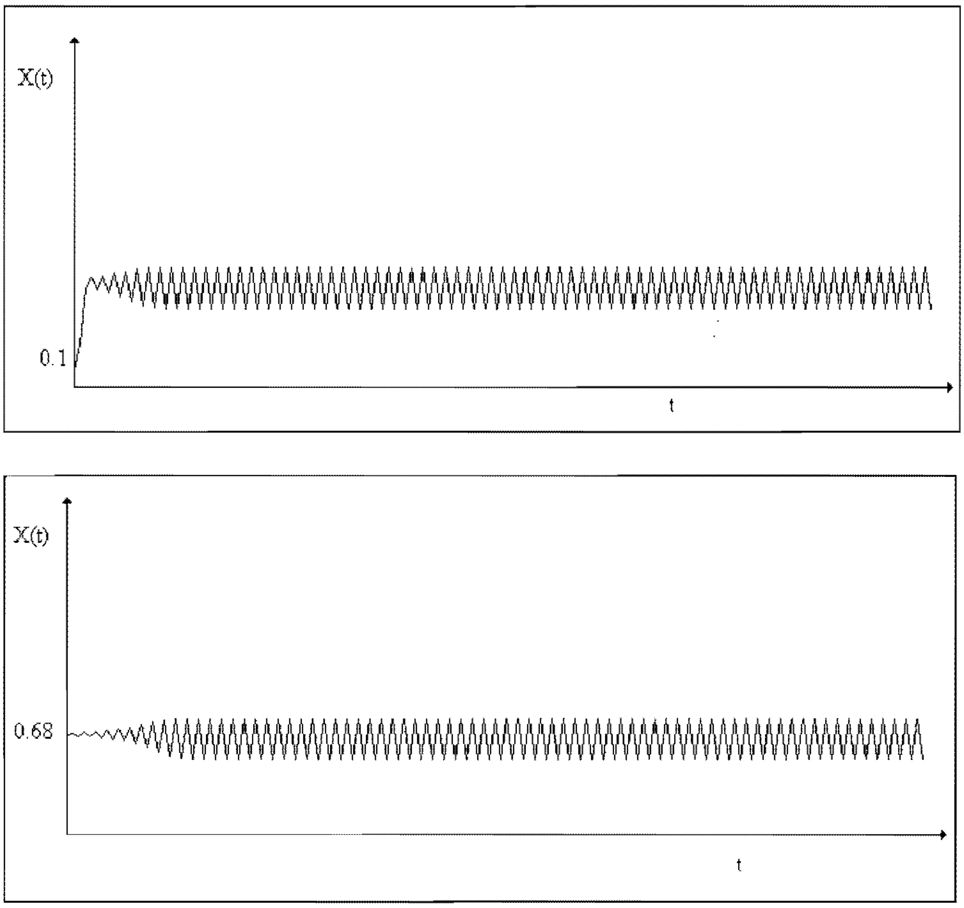


Figure 7: Time series plots of simulated realizations of the model in example (2) Starting with difference initial values x_0

2- Conclusion

In this paper, we study and find the stability conditions of logistic autoregressive model by using a dynamical approach due to Ozaki.

This approach bases on a local linearization technique near the non zero singular point of the model, by this technique all nonlinear autoregressive models approximate to a linear autoregressive models near the non zero singular point.

The first step of stability is the existence of a non-zero singular point that is if the model does not have a non-zero singular point, then the model is unstable. If the non-zero singular point exists and satisfies the stability conditions that we find, then the model is stable.

When the non-zero singular point is unstable, we search for existence of a limit cycles. If the limit cycle exists and stable, then the model is stable. Otherwise, it is unstable.

For a logistic autoregressive model, we find the stability conditions of the non-zero singular point of the model in terms of its parameter (a), when this parameter belongs to the interval [0, 4]. According to the stability conditions, this interval is divided into three subintervals.

In the first interval (when $0 \leq a \leq 1$) all trajectories approach to zero, that is zero is an attracting fixed point. (See [12])

In the second interval ($1<a<3$), the model is stable and all trajectories approach the non-zero singular point of the model.

In the third interval (3≤a≤4), the non-zero singular point of the model is unstable, but if the limit cycle exists, then it is stable if it satisfies the condition mentioned in proposition (1). Otherwise, it is unstable limit cycle.

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