Qatar Univ. Sci. Bull. (1983), 3: 7 - 11

ON 2n+1−VERTEX-FREE GRAPHS

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ABSTRACT

We shall introduce the concepts of pseudoadjacent, pseudocoatraction and pseudohomomorphism. We shall show, with the help of these concepts, in theorem 2 that every quadrilateral graph V (which is a 4-vertexgraph) has chromatic number 2. From this theorem follows our main result in theorem 4 that every 2n+1-vertex free graph has chromatic number 2.

All graphs considered in this paper are finite, without loops and multiple edges.

Definition 1 A planer graph (with at least 4 vertices) is called 2n+1−vertexfree graph when the regions in which it divides the plane are surrounded by elementary 2i-vertex ($i \geq 2$).

Definition 2 Two vertices in a quadrilateral graph V are said to be pseudoadjacent if they are situated diagonally on a quadrilateral.

Definition 3 A pseudocontraction in a quadrilateral graph V is a contraction of two pseudoadjacent vertices in V. (One can imagine such a contraction as follows: two vertices which are situated diagonally in a quadrilateral graph are connected by an edge and then contracted).

All contractions which take place in the following are pseudocontraction. For simplicity we shall call them contraction.

Lemma 1 Every quadrilateral graph V contains at least a vertex e with valency 2 or 3.

Proof: If the number of vertices of V is equal n then, because of Euler's formula, the number of its edges must be 2 . (n-2).

Let k_r be the number of vertices with valency r in V. Suppose that the valency of every vertex of V is ≥ 4.

Then:

$4k_4+5k_5+6k_6+\ldots = 2.2(n-2)$
$4(k_4+k_5+k_6+\ldots) + (k_5+2k_6+3k_7+\ldots) = 4(n-2)$
Let $k_5+2k_6+3k_7\ldots = K$.
Since: $k_4+k_5+k_6+\ldots = n$ it follows that:
$4n+K = 4n-8$ i.e. $K = -8$ which is a contradiction because K is a positive number.

Lemma 2 Let $Q_1 = (e_1, e_2, e_3, e_4)$ and $Q_2 = (e_1, e_2, e'_3, e'_4)$ with $Q_1 \cap Q_2 = (e_1, e_2)$ be two quadrilaterals in the quadrilateral graph V. If the edge (e_1, e_2) is deleted then it is possible to connect the two verices e'_3 and e_4 or e'_4 and e_3 by an edge so that the resulting graph is again a quadrilateral graph.

Proof: Only two of the vertex pairs e'_3 and e_4 or e'_4 and e_3 could be adjacent since all our graphs are planer. Suppose, without loss of generality that e'_3 and e_4 are adjacent in V. If the edge (e_1, e_2) is deleted from V then all the regions outside the region e_4, e_3, e_2, e'_3, e'_4, e_1 remain surrounded by quadrilaterals. If e'_4 and e_3 are connected now by an edge (e'_4, e_3) the region e_4, e_3, e_2, e'_3, e'_4, e_1 will be divided into two quadrilaterals. Therefore the resulting graph will again be quadrilateral graph.

Lemma 3 Let V be a quadrilateral graph with a vertex e which has valency 2 or 3. Then there is at least one contraction of e with a pseudo-adjacent vertex in V such that the resulting graph is again quadrilateral one.

Proof: Case 1 The valency $\gamma(e)$ of e is equal 2.

If a vertex p is added to a quadrilateral graph V such that p is connected to two pseudoadjacent vertices in V then the resulting graph is again a quadrilateral one. This is because through this operation one quadrilateral is simply divided into two quadrilaterals. All other quadrilaterals in V remain untouched.

If p is deleted together with the two edges the resulting graph is again quadrilateral, i.e. it is possible to add arbitrary vertices of valency 2 (and connect them with two (pseudoadjacent vertices) or to delete them without destroying the property quadrilateral graph. But the contraction of the vertex e with a pseudoadjacent vertex is the same as the deletion of e and the two edges incident on it.

Case 2 The valency (e) of e is equal 3. Let e_1, e_2 and e_3 be the three vertices which are adjacent to e in V. Every edge in V must lie on two surface quadrilaterals. Suppose that the edge (e, e_1) lies on the two quadrilaterals Q_1 and Q_2. Let $\{e_1, e, e_3, e''\}$ and $\{e_1, e, e_2, e'\}$ be vertex sets of Q_1 and Q_2 respectively.

Then:

a) It is possible, because of lemma 2, to delete the edge (e, e_1) in V and connect the vertices e_2 and e'' (or e_3 and e') in such a way that the resulting graph V′ is again quadrilateral graph. And b) $\gamma(e)$ is equal to two in V′. Then it is possible, because of case 1, to contract a and e'' with the resulting graph again quadrilateral. The two steps a) and b) carried on successively in V is equivalent to the pseudocontraction of e and e''. This proves our lemma.

Definition 4 Let A and B be two quadrilateral graphs. Then A is said to be pseudohomomorphic to B in symbols: $A > B$ if it is possible to get B from A through pseudocontractions in A.

Theorem 1 Let V_n and V_{n-1} be two quadrilateral graphs with number of vertices n and n-1 respectively. If $V_n > V_{n-1}$ and the chromatic number $\varnothing(V_{n-1})$ of V_{n-1} is equal 2 then: $\varnothing(Vn) = 2$

Proof: By lemma 1 the graph V must have at least a vertex e with valency $\gamma(e) = 2$ or 3.

Case 1 γ (e) = 2: Suppose that e_1, e_2 and e_3 are the other vertices of the quadrilateral on which e lies. Suppose without loss of generality that e and e_2 are situated diagonally and e and e_2 are not adjacent (otherwise we can consider e_1 and e_3 because V is planer). If e and e_2 are pseudocontracted the resulting graph V′ is again quadrilateral becauseof lemma 3. Since ∅(V) = 2 the two vertices e_1 and e_3 must have the same colour, say 1. The graph V can be obtained from V′ by adding a vertex e to V′ and connecting it to e_1 and e_3. Since e_1 and e_3 each has the colour 2 in V′ the vertex e can be given the colour 2.

Case 2 γ (e) = 3. Suppose that e_1, e_2 and e_3 are the vertices with which e is adjacent in V. Let Q_1 and Q_2 be the surface quadrilaterals in V which have the edge (e, e_1) in common. Let {e_1, e_1, e″, e_2} and {e_1, e_1, e′, e_3} be the vertex sets of Q_1 and Q_2 respectively. It is possible, because of Lemma 3, to contract e with at least one of the pseudoadjacent vertices e′ or e″ so that the resulting graph is quadrilateral one. Suppose, without lose of generality, we can contract e and e″. Let V′ be the resulting graph. Since ∅(V′) = 2 the two vertices e and e_2 must have the same colour, say 1 and the other two vertices e_1 and e_3 must have the same colour, say 2. In order to get the graph V back from V′ we need only to delete the edge (e_2, e″) in V′ and add the vertex e to V′ connecting it to the three vertices e_1, e_2 and e_3. Since e_1, e_2 and e_3 have all the same colour which is 1 in V′ it is possible to give the other colour 2 to the vertex e in V. This means that ∅(V) = ∅(V′) = 2.

Theorem 2 Every quadrilateral graph V has chromatic number 2.

Proof: Suppose n is the number of vertices of V. The theorem is true for n = 4 and so it is true for n−1. Because of lemma 3 there is a graph V′ with n−1 vertices such that: V > V′.

Since ∅(V′) = 2, because of induction hypothesis and V > V′; therefore (by theorem 2) ∅ (V) = 2.

Theorem 3 Every 2n+1-vertex free graph R has chromatic number 2.

Proof: Let {e_1, e_2, e_3,..., e_i} be the vertex set of one of the regions into which R divides the plane. Suppose that this region is surrounded by the set of edges {(e_1, e_2), (e_2, e_3), ..., (e_{i-1}, e_i)}. Suppose i is even and greater than 4. If i were equal to 4 for every region, R would be a quadrilateral graph and the theorem would follow immediately fom theorem 3. If we connect the vertex e_1 with e_4, e_5 with e_8, e_9 with e_{12}, ..., and e_{i-3} with ei we divide this region (without disturbing the other regions) into quadrilaterals. By repeating this process in the other regions which are surrounded by even number of edges greater than 4 we change the graph R in a quadrilateral graph V. It is possible to get the graph R back from V by simply deleting the edges which we have added to R. But the chromatic number ∅(V) of V remain unchanged by deleting edges from it. Therefore:
∅ (V) = ∅ (R) = 2

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حول الجراف الخالي من رؤوس درجتها ٢ ن + ١

قيس الوهابي

يجرى في هذا البحث تقديم فكرة شبه التجاور وشبه التشاكل وشبه الانكماش . وباستعمال هذه الأفكار يبرهن أن كل جراف رباعي له عدد كروماتيكي يساوى ٢ . ومن ذلك يبرهن أن كل جراف لا يحتوي على رؤوس من درجة ٢ ن + ١ له عدد كروماتيكي يساوي ٢