

Convergence theorems for nonself asymptotically nonexpansive mappings

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Abstract

In this paper, we prove some strong and weak convergence theorems using a modified iterative process for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space. This will improve and generalize the corresponding results in the existing literature. Finally, we will state that our theorems can be generalized to the case of finitely many mappings. © 2007 Elsevier Ltd. All rights reserved.

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1. Introduction

Let E be a real Banach space and C a nonempty subset of E . Let $S : C \rightarrow C$ be a self-mapping. Throughout this paper, we will denote the set of all positive integers. S is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that $\|S^n x - S^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. S is called uniformly k -Lipschitzian if for some $k > 0$, $\|S^n x - S^n y\| \leq k \|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in C$. S is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Asymptotically nonexpansive self-mappings using the Ishikawa iterative (a two-step iterative) and the Mann iterative (a one-step) processes have been studied by various authors. For example, see [1–3]. Glowinski and Le Tallec [4] applied a three-step iterative process for finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory.

Very recently, Suantai [5] introduced the following iterative process and used it for the weak and strong convergence of fixed points of self-mappings in a uniformly convex Banach space.

$$\begin{cases} x_1 = x \in C, \\ z_n = a_n T^n x_n + (1 - a_n) x_n, \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.1)$$

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where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$ satisfy certain conditions. It reduces to the Xu and Noor iterative process [6] for $c_n = \beta_n = 0$:

$$\begin{cases} x_1 = x \in C, \\ z_n = a_n T^n x_n + (1 - a_n)x_n, \\ y_n = b_n T^n z_n + (1 - b_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.2}$$

The Ishikawa iterative process [7] is obtained for $a_n = c_n = \beta_n = 0$:

$$\begin{cases} x_1 = x \in C, \\ y_n = b_n T^n x_n + (1 - b_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.3}$$

We get the Mann iterative process [8] for $a_n = b_n = c_n = \beta_n = 0$:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.4}$$

Recall that a subset C of E is called a retract of E if there exists a continuous map $P : E \rightarrow C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \rightarrow E$ is said to be a retraction if $P^2 = P$. It follows that if P is a retraction then $Py = y$ for all y in the range of P . Chidume et al. [9] defined nonself asymptotically nonexpansive mapping as follows.

Let $P : E \rightarrow C$ be a nonexpansive retraction of E into C . A nonself mapping $T : C \rightarrow E$ is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$, we have $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$. Also T is called uniformly k -Lipschitzian if for some $k > 0$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k \|x - y\|$ for all $n \in \mathbb{N}$ and all $x, y \in C$.

They studied the Mann iterative process for the case of nonself asymptotically nonexpansive mappings:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P \left(\alpha_n T(PT)^{n-1}x_n + (1 - \alpha_n)x_n \right), \quad n \in \mathbb{N}. \end{cases} \tag{1.5}$$

Inspired by (1.1) and (1.5), we give the following nonself version of (1.1):

$$\begin{cases} x_1 = x \in C, \\ z_n = P \left(a_n T(PT)^{n-1}x_n + (1 - a_n)x_n \right), \\ y_n = P \left(b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1 - b_n - c_n)x_n \right), \\ x_{n+1} = P \left(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n + (1 - \alpha_n - \beta_n)x_n \right), \end{cases} \tag{1.6}$$

for all $n \in \mathbb{N}$.

Clearly, we can obtain the corresponding nonself versions of (1.2)–(1.4). We shall obtain the strong and weak convergence theorems using (1.6) for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space. As remarked earlier, Suantai [5] has established weak and strong convergence criteria for asymptotically nonexpansive self-mappings while Chidume et al. [9] studied the Mann iterative process for the case of nonself mappings. Our results will thus improve and generalize corresponding results of Suantai [5] and others for nonself mappings and those of Chidume et al. [9] in the sense that our iterative process contains the one used by them.

2. Preliminaries

Let E be a real Banach space and let C be a nonempty closed convex subset of E . A mapping $T : C \rightarrow E$ is called demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightarrow x$ (weak convergence to x) and $Tx_n \rightarrow y$ imply that $x \in C$ and $Tx = y$. We need the following lemmas.

Lemma 1 ([9]). Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$ as $n \rightarrow \infty$. Then $I - T$ is demiclosed at zero.

Lemma 2 ([10]). Let $\{r_n\}, \{s_n\}$ and $\{t_n\}$ be nonnegative sequences satisfying

$$r_{n+1} \leq (1 + s_n)r_n + t_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \rightarrow \infty} r_n$ exists. Moreover, if $\liminf_{n \rightarrow \infty} r_n = 0$, then $\lim_{n \rightarrow \infty} r_n = 0$.

The following characterization of a uniformly convex Banach space proved by Xu [11] will be used.

Lemma 3. Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda)\|y\|^p - \omega_p(\lambda)g(\|x - y\|) \tag{2.1}$$

for all $x, y \in U$ and $0 \leq \lambda \leq 1$ where U is a unit ball of radius r centered at 0 and

$$\omega_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p.$$

In particular, for $p = 2$, (2.1) becomes

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|). \tag{2.2}$$

Lemma 4 ([12]). Let E be a uniformly convex Banach space and $B_r = \{x \in E : \|x\| \leq r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \lambda\beta g(\|x - y\|) \tag{2.3}$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

3. Convergence theorems

Lemma 5. Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose further that the set $F(T)$ of fixed points of T is nonempty. Define a sequence $\{x_n\}$ in C as in (1.6) where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ are such that $b_n + c_n$ and $\alpha_n + \beta_n$ remain in $[0, 1]$. Then we have the following:

- (1) If $w \in F(T)$, then $\lim_{n \rightarrow \infty} \|x_n - w\|$ exists.
- (2) If $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n) < 1$ and $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$, then $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.

Proof. Let q be a fixed point of T . Then by (1.6) and Lemma 3, we have

$$\begin{aligned} \|z_n - q\|^2 &= \left\| P(a_n T(PT)^{n-1} x_n + (1 - a_n)x_n) - Pq \right\|^2 \\ &\leq \left\| a_n T(PT)^{n-1} x_n + (1 - a_n)x_n - q \right\|^2 \\ &= \left\| a_n(T(PT)^{n-1} x_n - q) + (1 - a_n)(x_n - q) \right\|^2 \\ &\leq a_n \left\| T(PT)^{n-1} x_n - q \right\|^2 + (1 - a_n) \|x_n - q\|^2 - \omega_2(a_n) g_1 \left(\left\| T(PT)^{n-1} x_n - x_n \right\| \right) \\ &\leq a_n \left\| T(PT)^{n-1} x_n - q \right\|^2 + (1 - a_n) \|x_n - q\|^2 \\ &\leq a_n k_n^2 \|x_n - q\|^2 + (1 - a_n) \|x_n - q\|^2 \\ &= \left(1 - a_n + a_n k_n^2 \right) \|x_n - q\|^2. \end{aligned}$$

Now by (1.6) and Lemma 4, we have

$$\begin{aligned} \|y_n - q\|^2 &= \left\| P \begin{pmatrix} b_n T(PT)^{n-1} z_n + c_n T(PT)^{n-1} x_n \\ + (1 - b_n - c_n)x_n \end{pmatrix} - Pq \right\|^2 \\ &\leq \left\| \begin{pmatrix} b_n T(PT)^{n-1} z_n + c_n T(PT)^{n-1} x_n \\ + (1 - b_n - c_n)x_n - q \end{pmatrix} \right\|^2 \\ &= \left\| \begin{pmatrix} b_n (T(PT)^{n-1} z_n - q) + c_n (T(PT)^{n-1} x_n - q) \\ + (1 - b_n - c_n)(x_n - q) \end{pmatrix} \right\|^2 \\ &\leq b_n \|T(PT)^{n-1} z_n - q\|^2 + c_n \|T(PT)^{n-1} x_n - q\|^2 \\ &\quad + (1 - b_n - c_n) \|x_n - q\|^2 - b_n(1 - b_n - c_n)g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \\ &\leq b_n k_n^2 \|z_n - q\|^2 + c_n k_n^2 \|x_n - q\|^2 \\ &\quad + (1 - b_n - c_n) \|x_n - q\|^2 - b_n(1 - b_n - c_n)g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right). \end{aligned}$$

Moreover,

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \left\| P \begin{pmatrix} \alpha_n T(PT)^{n-1} y_n + \beta_n T(PT)^{n-1} z_n \\ + (1 - \alpha_n - \beta_n)x_n \end{pmatrix} - Pq \right\|^2 \\ &\leq \alpha_n \|T(PT)^{n-1} y_n - q\|^2 + \beta_n \|T(PT)^{n-1} z_n - q\|^2 \\ &\quad + (1 - \alpha_n - \beta_n) \|x_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n)g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\ &\leq \alpha_n k_n^2 \|y_n - q\|^2 + \beta_n k_n^2 \|z_n - q\|^2 \\ &\quad + (1 - \alpha_n - \beta_n) \|x_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n)g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\ &\leq \alpha_n k_n^2 \left(\begin{matrix} b_n k_n^2 \|z_n - q\|^2 + c_n k_n^2 \|x_n - q\|^2 \\ + (1 - b_n - c_n) \|x_n - q\|^2 \\ - b_n(1 - b_n - c_n)g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \end{matrix} \right) \\ &\quad + \beta_n k_n^2 \|z_n - q\|^2 + (1 - \alpha_n - \beta_n) \|x_n - q\|^2 \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\ &\leq \|x_n - q\|^2 + \left(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1 - b_n - c_n) - \alpha_n - \beta_n \right) \|x_n - q\|^2 \\ &\quad + \left(\alpha_n b_n k_n^4 + \beta_n k_n^2 \right) \|z_n - q\|^2 - \alpha_n b_n k_n^2(1 - b_n - c_n)g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\ &\leq \|x_n - q\|^2 + \left(\alpha_n c_n k_n^2(k_n^2 - 1) + \alpha_n(k_n^2 - 1) - \alpha_n k_n^2 b_n - \beta_n \right) \|x_n - q\|^2 \\ &\quad + \left(\alpha_n b_n k_n^4 + \beta_n k_n^2 \right) \left(1 + \alpha_n k_n^2 - \alpha_n \right) \|x_n - q\|^2 \\ &\quad - \alpha_n b_n k_n^2(1 - b_n - c_n)g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \\ &\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\ &= \|x_n - q\|^2 + \left(\begin{matrix} \alpha_n c_n k_n^2(k_n^2 - 1) + \alpha_n(k_n^2 - 1) - \alpha_n k_n^2 b_n - \beta_n \\ \alpha_n b_n k_n^4 + \beta_n k_n^2 + (\alpha_n b_n k_n^4 + \beta_n k_n^2) \alpha_n(k_n^2 - 1) \end{matrix} \right) \|x_n - q\|^2 \end{aligned}$$

$$\begin{aligned}
 & -\alpha_n b_n k_n^2 (1 - b_n - c_n) g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \\
 & -\alpha_n (1 - \alpha_n - \beta_n) g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\
 = & \|x_n - q\|^2 + \left(\begin{aligned} & \alpha_n c_n k_n^2 (k_n^2 - 1) + \alpha_n (k_n^2 - 1) + \alpha_n k_n^2 b_n (k_n^2 - 1) \\ & + \beta_n (k_n^2 - 1) + (\alpha_n b_n k_n^4 + \beta_n k_n^2) \alpha_n (k_n^2 - 1) \end{aligned} \right) \|x_n - q\|^2 \\
 & -\alpha_n b_n k_n^2 (1 - b_n - c_n) g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \\
 & -\alpha_n (1 - \alpha_n - \beta_n) g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\
 = & \|x_n - q\|^2 + (k_n^2 - 1) \left(\begin{aligned} & \alpha_n c_n k_n^2 + \alpha_n + \alpha_n k_n^2 b_n \\ & + \beta_n + (\alpha_n b_n k_n^4 + \beta_n k_n^2) \alpha_n \end{aligned} \right) \|x_n - q\|^2 \\
 & -\alpha_n b_n k_n^2 (1 - b_n - c_n) g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \\
 & -\alpha_n (1 - \alpha_n - \beta_n) g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \\
 \leq & \|x_n - q\|^2 + (k_n^2 - 1) M - \alpha_n b_n k_n^2 (1 - b_n - c_n) g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \\
 & -\alpha_n (1 - \alpha_n - \beta_n) g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right)
 \end{aligned}$$

for some $M > 0$. From the last inequality, we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + (k_n^2 - 1) M, \tag{3.1}$$

$$\alpha_n b_n k_n^2 (1 - b_n - c_n) g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + (k_n^2 - 1) M \tag{3.2}$$

and

$$\alpha_n b_n k_n^2 (1 - \alpha_n - \beta_n) g_2 \left(\|T(PT)^{n-1} y_n - x_n\| \right) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + (k_n^2 - 1) M. \tag{3.3}$$

Now from (3.1) and Lemma 2, it is clear that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists and the first part of lemma is over.

Next, we prove that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. By the conditions $\liminf_{n \rightarrow \infty} \alpha_n > 0$ and $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n) < 1$, there exist a positive integer n_0 and $\delta, \delta' \in (0, 1)$ such that $0 < \delta < \alpha_n, 0 < \delta < b_n$ and $b_n + c_n < \delta' < 1$ for all $n \geq n_0$. Thus from (3.2), we have

$$\delta^2 (1 - \delta') g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + (k_n^2 - 1) M$$

for all $n \geq n_0$.

So for $m \geq n_0$, we can write

$$\begin{aligned}
 \sum_{n=n_0}^m g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) & \leq \frac{1}{\delta^2 (1 - \delta')} \left(\sum_{n=n_0}^m (\|x_n - q\|^2 - \|x_{n+1} - q\|^2) + M \sum_{n=n_0}^m (k_n^2 - 1) \right) \\
 & \leq \frac{1}{\delta^2 (1 - \delta')} \left(\|x_{n_0} - q\|^2 + M \sum_{n=n_0}^m (k_n^2 - 1) \right).
 \end{aligned}$$

Letting $m \rightarrow \infty$, we have $\sum_{n=n_0}^{\infty} g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) < \infty$ so that $\lim_{n \rightarrow \infty} g_2 \left(\|T(PT)^{n-1} z_n - x_n\| \right) = 0$ which by continuity of g_2 implies that

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-1} z_n - x_n\| = 0. \tag{3.4}$$

Similarly,

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-1} y_n - x_n\| = 0. \tag{3.5}$$

To prove $\lim_{n \rightarrow \infty} \|T(PT)^{n-1}x_n - x_n\| = 0$, first consider

$$\begin{aligned} \|y_n - x_n\| &= \left\| P \left(\begin{matrix} b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n \\ + (1 - b_n - c_n)x_n \end{matrix} \right) - Px_n \right\| \\ &\leq \left\| b_n \left(T(PT)^{n-1}z_n - x_n \right) + c_n \left(T(PT)^{n-1}x_n - x_n \right) \right\| \\ &\leq b_n \left\| T(PT)^{n-1}z_n - x_n \right\| + c_n \left\| T(PT)^{n-1}x_n - x_n \right\|. \end{aligned} \tag{3.6}$$

Then

$$\begin{aligned} \left\| T(PT)^{n-1}x_n - x_n \right\| &\leq \left\| T(PT)^{n-1}x_n - T(PT)^{n-1}y_n \right\| + \left\| T(PT)^{n-1}y_n - x_n \right\| \\ &\leq k_n \|x_n - y_n\| + \left\| T(PT)^{n-1}y_n - x_n \right\| \\ &\leq k_n b_n \left\| T(PT)^{n-1}z_n - x_n \right\| + k_n c_n \left\| T(PT)^{n-1}x_n - x_n \right\| + \left\| T(PT)^{n-1}y_n - x_n \right\|. \end{aligned}$$

This yields

$$(1 - k_n c_n) \left\| T(PT)^{n-1}x_n - x_n \right\| \leq k_n b_n \left\| T(PT)^{n-1}z_n - x_n \right\| + \left\| T(PT)^{n-1}y_n - x_n \right\|.$$

Since $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n) < 1$ so there exist a $\gamma \in (0, 1)$ and a positive integer n_0 such that

$$\left\| T(PT)^{n-1}x_n - x_n \right\| \leq \frac{\gamma}{1 - \gamma} \left\| T(PT)^{n-1}z_n - x_n \right\| + \frac{1}{1 - \gamma} \left\| T(PT)^{n-1}y_n - x_n \right\|$$

for all $n \geq n_0$. Now with the help of (3.4) and (3.5), we have

$$\lim_{n \rightarrow \infty} \left\| T(PT)^{n-1}x_n - x_n \right\| = 0. \tag{3.7}$$

A joint effect of (3.4) and (3.7) on (3.6) provides

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{3.8}$$

Also note that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \left\| P \left(\begin{matrix} \alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n \\ + (1 - \alpha_n - \beta_n)x_n \end{matrix} \right) - Px_n \right\| \\ &\leq \alpha_n \left\| T(PT)^{n-1}y_n - x_n \right\| + \beta_n \left\| T(PT)^{n-1}z_n - x_n \right\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

so that

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.9}$$

Furthermore, from

$$\left\| x_{n+1} - T(PT)^{n-1}y_n \right\| \leq \|x_{n+1} - x_n\| + \left\| x_n - T(PT)^{n-1}y_n \right\|,$$

we find that

$$\lim_{n \rightarrow \infty} \left\| x_{n+1} - T(PT)^{n-1}y_n \right\| = 0. \tag{3.10}$$

Finally, we make use of the fact that every asymptotically nonexpansive mapping is uniformly L -Lipschitzian which when combined with (3.7), (3.9) and (3.10) gives

$$\begin{aligned} \|x_n - Tx_n\| &\leq \left\| x_n - T(PT)^{n-1}x_n \right\| + \left\| T(PT)^{n-1}x_n - T(PT)^{n-1}y_{n-1} \right\| + \left\| T(PT)^{n-1}y_{n-1} - Tx_n \right\| \\ &\leq \left\| x_n - T(PT)^{n-1}x_n \right\| + k_n \|x_n - y_{n-1}\| + L \left\| T(PT)^{n-2}y_{n-1} - x_n \right\| \end{aligned}$$

so that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad \square \tag{3.11}$$

We are now in a position to prove our first strong convergence theorem as follows.

Theorem 1. *Let E, C, T and $\{x_n\}$ be as in Lemma 5. If, in addition, T is either completely continuous or demicompact and $F(T) \neq \phi$, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T .*

Proof. Since T is completely continuous and $\{x_n\} \subseteq C$, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\lim_{k \rightarrow \infty} Tx_{n_k} = q^*$ (say). Therefore by (3.11), $x_{n_k} \rightarrow q^*$ as $k \rightarrow \infty$. By continuity of T , $Tq^* = q^*$. Moreover, as $\lim_{n \rightarrow \infty} \|x_n - q^*\|$ exists for all $q^* \in F(T)$, therefore $\{x_n\}$ converges strongly to q^* . That is,

$$\lim_{n \rightarrow \infty} \|x_n - q^*\| = 0. \tag{3.12}$$

Hence

$$\|y_n - q^*\| \leq \|y_n - x_n\| + \|x_n - q^*\|$$

implies with the help of (3.8) and (3.12) that $\{y_n\}$ converges strongly to a fixed point q^* of T . Another simple argument proves that $\{z_n\}$ converges strongly to a fixed point q^* of T .

Next, assume that T is demicompact. Since $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \rightarrow \infty} x_{n_k} = q'$ (say). By Lemma 1, $q' = Tq'$. Moreover, as $\lim_{n \rightarrow \infty} \|x_n - q^*\|$ exists for all $q^* \in F(T)$, therefore $\{x_n\}$ converges strongly to q' . That is,

$$\lim_{n \rightarrow \infty} \|x_n - q'\| = 0.$$

An argument similar to the above case proves that $\{y_n\}$ and $\{z_n\}$ also converge strongly to a fixed point q' of T . This completes the proof. \square

Theorem 2. *Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}$ and $\{\alpha_n\}$ be in $[0, 1]$ such that $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$ and $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C as*

$$\begin{aligned} x_1 &= x \in C, \\ z_n &= P \left(a_n T(PT)^{n-1} x_n + (1 - a_n) x_n \right), \\ y_n &= P \left(b_n T(PT)^{n-1} z_n + (1 - b_n) x_n \right), \\ x_{n+1} &= P \left(\alpha_n T(PT)^{n-1} y_n + (1 - \alpha_n) x_n \right), \quad n \in \mathbb{N}. \end{aligned}$$

If T is completely continuous and $F(T) \neq \phi$, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T .

Proof. Put $c_n = \beta_n = 0$ in Theorem 1 to get the result. \square

Remark 1. If T is a self-mapping then Theorem 1 generalizes Theorem 2.3 of Suantai [5]. Also note that we have not imposed the condition of boundedness on C as opposed to [5]. By the same argument, Theorem 2 is a generalization of Theorem 2.4 of Suantai [5] and Theorem 2.1 of Xu and Noor [6].

The following theorem generalizes Theorem 2.5 of Suantai [5] and Theorem 3 of Rhoades [13].

Theorem 3. *Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{b_n\}$ and $\{\alpha_n\}$ be in $[0, 1]$ be such that $0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} b_n < 1$ and $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq$*

$\limsup_{n \rightarrow \infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C as

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= P\left(b_n T(PT)^{n-1} x_n + (1 - b_n) x_n\right), \\ x_{n+1} &= P\left(\alpha_n T(PT)^{n-1} y_n + (1 - \alpha_n) x_n\right), \quad n \in \mathbb{N}. \end{aligned}$$

If T is completely continuous and $F(T) \neq \phi$, then $\{x_n\}$ and $\{y_n\}$ converge strongly to a fixed point of T .

Proof. The choice $a_n = c_n = \beta_n = 0$ in Theorem 1 leads to the conclusion. \square

Theorem 2.2 of Schu [3], Theorem 2.6 of Suantai [5], Theorem 2 of Rhoades [13] and Theorem 1.5 of Schu [14] have been generalized as in the following:

Theorem 4. Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}$ in $[0, 1]$ be such that $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C as

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P\left(\alpha_n T(PT)^{n-1} x_n + (1 - \alpha_n) x_n\right), \quad n \in \mathbb{N}. \end{aligned}$$

If T is completely continuous and $F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a fixed point of T .

Proof. Put $a_n = b_n = c_n = \beta_n = 0$ in Theorem 1. \square

In the same way, we can prove Lemma 5 under the conditions used by Chidume et al. [9] to get the following:

Theorem 5. Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Define a sequence $\{x_n\}$ in C as in (1.6) where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{b_n + c_n\}, \{\alpha_n + \beta_n\}$ are in $[\epsilon, 1 - \epsilon]$ for all $n \geq 1$ and for some ϵ in $(0, 1)$. If T is completely continuous and $F(T) \neq \phi$, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converges strongly to a fixed point of T .

This theorem immediately gives the following:

Corollary 1 ([9, Theorem 3.7]). Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{\alpha_n\}$ in $(0, 1)$ be such that $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \geq 1$ and for some ϵ in $(0, 1)$. Define a sequence $\{x_n\}$ in C as

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= P\left(\alpha_n T(PT)^{n-1} x_n + (1 - \alpha_n) x_n\right), \quad n \in \mathbb{N}. \end{aligned}$$

If T is completely continuous and $F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a fixed point of T .

Now we turn our attention towards weak convergence. A Banach space E is said to satisfy Opial's condition [15] if for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x$ implies that $\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Actually, if T is not taken to be completely continuous but E satisfies Opial's condition, then we have the following:

Theorem 6. Let E be a uniformly convex Banach space satisfying Opial's condition and let C be its closed and convex subset. Let $T : C \rightarrow E$ be a nonself asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be in $[0, 1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$. Define a sequence $\{x_n\}$ in C as in (1.6). If $F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of T .

Proof. Let $q \in F(T)$. Then as proved in Lemma 5, $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 5, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed at zero by Lemma 1, therefore we obtain $Tz_1 = z_1$. In the same way, we can prove that $z_2 \in F(T)$. Next, we prove the uniqueness. For this suppose that $z_1 \neq z_2$, then by Opial's condition

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_1\| &= \lim_{n_i \rightarrow \infty} \|x_{n_i} - z_1\| \\ &< \lim_{n_i \rightarrow \infty} \|x_{n_i} - z_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_2\| \\ &= \lim_{n_j \rightarrow \infty} \|x_{n_j} - z_2\| \\ &< \lim_{n_j \rightarrow \infty} \|x_{n_j} - z_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_1\|. \end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in $F(T)$. \square

Remark 2. The above Theorem contains Theorem 2.8, Corollaries 2.9–2.11 of Suantai [5] as special cases when T is a self-mapping.

4. Finitely many mappings case

Nothing prevents one from proving the results of the previous section for finitely many mappings case. However, we just state the case of three mappings. Thus one can easily prove the following.

Theorem 7. Let E be a uniformly convex Banach space and let C be its closed and convex subset. Let $T_1, T_2, T_3 : C \rightarrow E$ be three nonself asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose further that the set F of common fixed points of $T_i, i = 1, 2, 3$ is nonempty. Define a sequence $\{x_n\}$ in C as:

$$\begin{cases} x_1 = x \in C, \\ z_n = P \left(a_n T_3 (P T_3)^{n-1} x_n + (1 - a_n) x_n \right), \\ y_n = P \left(b_n T_2 (P T_2)^{n-1} z_n + c_n T_2 (P T_2)^{n-1} x_n + (1 - b_n - c_n) x_n \right), \\ x_{n+1} = P \left(\alpha_n T_1 (P T_1)^{n-1} y_n + \beta_n T_1 (P T_1)^{n-1} z_n + (1 - \alpha_n - \beta_n) x_n \right), \quad n \in \mathbb{N}. \end{cases}$$

Suppose either

- (1) $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ which satisfy: $b_n + c_n \in [0, 1], \alpha_n + \beta_n \in [0, 1], 0 < \liminf_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} (b_n + c_n) < 1$ and $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n)$, or
- (2) $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are sequences in $[\epsilon, 1 - \epsilon]$ where $\epsilon \in (0, 1)$.

If one of the $T_i, i = 1, 2, 3$ is either completely continuous or demicompact and $F \neq \emptyset$, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a point of F .

- Remark 3.** (1) The above theorem when reduced to two mappings case contains Theorems 3.3 and 3.4 of Wang [16].
 (2) The above theorem when extended to finitely many mappings case contains Theorems 3.4 and 4.1 of Chidume and Ali [17].

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