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A best proximity point theorem for special generalized proximal β-quasi contractive mappings

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Abstract

In this paper, we obtain some best proximity point results for a new class of non-self mappings $T : A \longrightarrow B$ called special generalized proximal β-quasi contractive. Our result is illustrated by an example. Several consequences are derived.

MSC: 47H10; 54H25

Keywords: Best proximity points; Special generalized proximal β-quasi contractive mappings

1 Introduction

The famous Banach contraction principle guarantees the existence and uniqueness of fixed points of self-mappings $T : X \longrightarrow X$, where (X, d) is a complete metric space. The Banach contraction principle has been generalized in different ways as in [1]. The main interesting studies deal with the extension of Banach's contraction to non-self-mappings $T : A \longrightarrow B$, where (A, B) is a pair of subsets of a metric space (X, d). In fact such mappings do not necessarily have fixed points. The idea is to look for points where $d(\zeta, T\zeta) = d(A, B)$. Such points are called best proximity points. In 1969, a best approximation theorem was introduced by Fan [2]. Later on, Sadiq Basha [3] proposed necessary and sufficient conditions for the existence of proximal contractions of first and second kind for such points. Several variants of non-self-contractions for the existence of a best proximity point were studied in [4–7].

In 2014, Almeida et al. [8], by using the notion of P-property (weak P-property), proved that some late results about the existence and uniqueness of best proximity points can be obtained from the versions of associated existing results in the fixed point theory.

Our work focuses on the best proximity point theorem for a new family of non-self-mappings called special generalized proximal β-quasi contractive mappings. As an application to the self-mapping case, the present work generalizes several existing results on fixed point theory as the Banach contraction principle [9] and the generalization of such a principle by Ćirić in [1].

The paper is divided into five sections. Section 2 introduces the notation used herein, presents some definitions, and recalls some useful results. The best proximity point the-

orem with its proof is stated in Sect. 3. Finally, several consequences on the existence and uniqueness of best proximity points and fixed point results are given in Sect. 4.

2 Preliminaries and definitions

Let (A, B) be a pair of nonempty subsets of a metric space (X, d). Throughout this work we consider the following notations:

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\};$$

$$A_0 := \{a \in A : \text{there exists } b \in B \text{ such that } d(a, b) = d(A, B)\};$$

$$B_0 := \{b \in B : \text{there exists } a \in A \text{ such that } d(a, b) = d(A, B)\}.$$

Definition 2.1 ([3]) Let $T : A \to B$ be a mapping. An element x^* is said to be a best proximity point of T if $d(x^*, Tx^*) = d(A, B)$.

Definition 2.2 ([10]) Let $\beta \in (0, +\infty)$. A β-comparison function is a map $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:
  (1) φ is nondecreasing;
  (2) $\lim_{n \to \infty} \varphi_\beta^n(t) = 0$ for all $t > 0$, where φ_β^n denotes the nth iterate of φ_β and
       $\varphi_\beta(t) = \varphi(\beta t)$;
  (3) there exists $s \in (0, +\infty)$ such that $\sum_{n=1}^{\infty} \varphi_\beta^n(s) < \infty$.
The set of all β-comparison functions φ satisfying (1), (2), and (3) will be denoted by Φ_β.

Remark 2.3 Let $\alpha, \beta \in (0, +\infty)$. If $\alpha < \beta$, then $\Phi_\beta \subset \Phi_\alpha$.

A useful lemma concerning the comparison functions Φ_β was performed in [10].

Lemma 2.4 ([10]) *Let $\beta \in (0, +\infty)$ and $\varphi \in \Phi_\beta$. Then*
  (1) φ_β *is nondecreasing*;
  (2) $\varphi_\beta(t) < t$ *for all $t > 0$*;
  (3) $\sum_{n=1}^{\infty} \varphi_\beta^n(t) < \infty$ *for all $t > 0$.*

Definition 2.5 ([11]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) such that A_0 is nonempty. Then the pair (A, B) is said to have the P-property iff $d(\zeta_1, \eta_1) = d(\zeta_2, \eta_2) = d(A, B) \Longrightarrow d(\zeta_1, \zeta_2) = d(\eta_1, \eta_2)$, where $\zeta_1, \zeta_2 \in A$ and $\eta_1, \eta_2 \in B$.

Definition 2.6 We say that B is approximately compact with respect to A iff every sequence $\{\eta_n\} \subset B$ satisfying $\lim_{n \to +\infty} d(\zeta, \eta_n) = d(\zeta, B)$ for some $\zeta \in A$ has a convergent subsequence.

3 Main results and theorems

First, we introduce the following concept.

Definition 3.1 Let (X, d) be a metric space and (A, B) be a pair of nonempty subsets of X. Let $\beta \in (0, +\infty)$. A non-self-mapping $T : A \to B$ is said to be a special generalized proximal β-quasi contractive mapping iff there exist $\varphi \in \Phi_\beta$ and positive numbers α_i, $i = 1, 2, 3, 4$, such that:

$$d(T\zeta, T\eta) \leq \varphi\big(M_T(\zeta, \eta)\big), \quad \forall \zeta, \eta \in A, \tag{3.1}$$

where

$$M_T(\zeta, \eta) = \max\big\{\alpha_0 d(\zeta, \eta), \alpha_1\big(d(\zeta, T\zeta) - d(A, B)\big),$$
$$\alpha_2\big(d(\eta, T\eta) - d(A, B)\big), \alpha_3\big(d(\eta, T\zeta) - d(A, B)\big),$$
$$\alpha_4\big(d(\zeta, T\eta) - d(A, B)\big)\big\}.$$

$M_T(\zeta, \eta)$ was introduced in [12]. Our main result is given by the following best proximity point theorem.

Theorem 3.2 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Consider a non-self-mapping $T : A \longrightarrow B$ satisfying the following conditions*:
  (1) *$T(A_0) \subset B_0$ and the pair (A, B) satisfies the P-property.*
  (2) *B is approximately compact with respect to A.*
  (3) *There exist elements $\zeta_0, \zeta_1 \in A$ such that*

$$d(\zeta_1, T\zeta_0) = d(A, B).$$

  (4) *There exists $\beta \geq \max_{0 \leq k \leq 3}\{\alpha_k, 2\alpha_4\}$ such that T is special generalized proximal β-quasi contractive.*
Moreover, assume that one of the following conditions holds:
  • *φ is continuous*;
  • *$\beta > \max\{\alpha_1, \alpha_3\}$.*
Then T has a unique best proximity point $\zeta \in A$ such that $d(\zeta, T\zeta) = d(A, B)$.

Proof From condition (3), there exist $\zeta_0, \zeta_1 \in A$ such that $d(\zeta_1, T\zeta_0) = d(A, B)$. Then $A_0 \neq \phi$, $B_0 \neq \phi$. Now $T(\zeta_1) \in B_0$ since $T(A_0) \subset B_0$, so there exists $\zeta_2 \in A$ such that $d(\zeta_2, T\zeta_1) = d(A, B)$. Continuing this process, we can build a sequence $\{\zeta_n\} \subset A_0$ such that

$$d(\zeta_{n+1}, T\zeta_n) = d(A, B) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Using the P-property, we get

$$d(\zeta_n, \zeta_{n+1}) = d(T\zeta_{n-1}, T\zeta_n). \tag{3.3}$$

Our next step is to prove that $\{\zeta_n\}$ is a Cauchy sequence.

Since T is special generalized proximal β-quasi contractive, we obtain

$$d(\zeta_n, \zeta_{n+1}) = d(T\zeta_{n-1}, T\zeta_n) \leq \varphi\big(M_T(\zeta_{n-1}, \zeta_n)\big), \quad \forall n \in \mathbb{N}. \tag{3.4}$$

On the other hand, using (3.2), (3.3), and the triangular inequality, we get

$$M_T(\zeta_{n-1}, \zeta_n) = \max\big\{\alpha_0 d(\zeta_{n-1}, \zeta_n), \alpha_1\big(d(\zeta_{n-1}, T\zeta_{n-1}) - d(A, B)\big),$$
$$\alpha_2\big(d(\zeta_n, T\zeta_n) - d(A, B)\big), \alpha_3\big(d(\zeta_n, T\zeta_{n-1}) - d(A, B)\big),$$
$$\alpha_4\big(d(\zeta_{n-1}, T\zeta_n) - d(A, B)\big)\big\}$$

$$\begin{aligned}
&= \max\big\{\alpha_0 d(\zeta_{n-1},\zeta_n), \alpha_1\big(d(\zeta_{n-1}, T\zeta_{n-1}) - d(A,B)\big),\\
&\qquad \alpha_2\big(d(\zeta_n, T\zeta_n) - d(A,B)\big), \alpha_4\big(d(\zeta_{n-1}, T\zeta_n) - d(A,B)\big)\big\}\\
&\leq \max\big\{\alpha_0 d(\zeta_{n-1},\zeta_n), \alpha_1 d(\zeta_{n-1},\zeta_n), \alpha_2 d(\zeta_n,\zeta_{n+1}),\\
&\qquad \alpha_4 d(\zeta_{n-1},\zeta_n) + \alpha_4 d(\zeta_n,\zeta_{n+1})\big\}\\
&\leq \beta \max\big\{d(\zeta_{n-1},\zeta_n), d(\zeta_n,\zeta_{n+1})\big\}.
\end{aligned}$$

Hence,

$$M_T(\zeta_{n-1},\zeta_n) \leq \beta \max\big\{d(\zeta_{n-1},\zeta_n), d(\zeta_n,\zeta_{n+1})\big\}, \tag{3.5}$$

where $\beta \geq \max_{0 \leq k \leq 3}\{\alpha_k, 2\alpha_4\}$. Using inequalities (3.3) and (3.5) and taking into consideration the fact that φ is nondecreasing, we get that

$$d(\zeta_{n+1},\zeta_n) \leq \varphi\big(\beta \max\big\{d(\zeta_{n-1},\zeta_n), d(\zeta_n,\zeta_{n+1})\big\}\big) = \varphi_\beta\big(\max\big\{d(\zeta_{n-1},\zeta_n), d(\zeta_n,\zeta_{n+1})\big\}\big).$$

Suppose that, for some n, we have $d(\zeta_{n-1},\zeta_n) \leq d(\zeta_n,\zeta_{n+1})$. It follows that $d(\zeta_{n+1},\zeta_n) \leq \varphi_\beta(d(\zeta_{n+1},\zeta_n)) < d(\zeta_{n+1},\zeta_n)$, which is a contradiction.

Then, for all $n \geq 0$, we necessary have $d(\zeta_{n-1},\zeta_n) > d(\zeta_n,\zeta_{n+1})$, and it follows that

$$d(\zeta_{n+1},\zeta_n) \leq \varphi_\beta\big(d(\zeta_{n-1},\zeta_n)\big), \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Then, by induction, we obtain that

$$d(\zeta_{n+1},\zeta_n) \leq \varphi_\beta^n\big(d(\zeta_1,\zeta_0)\big), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.7}$$

Let $\epsilon > 0$ be fixed. Since the numerical series $\sum_{n=1}^{+\infty} \varphi_\beta^n(d(\zeta_1,\zeta_0))$ converges, there exists a positive integer N such that $\sum_{n \geq N}^{+\infty} \varphi_\beta^n(d(\zeta_1,\zeta_0)) < \epsilon$. For $m > n > N$, using the triangular inequality, the convergence of the series, and (3.7), we obtain

$$\begin{aligned}
d(\zeta_n,\zeta_m) &\leq \sum_{k=n}^{m-1} d(\zeta_k,\zeta_{k+1})\\
&\leq \sum_{k=n}^{m-1} \varphi_\beta^k\big(d(\zeta_1,\zeta_0)\big)
\end{aligned}$$

since the series $\sum_{n=1}^{+\infty} \varphi_\beta^n(t)$ converges for all $t \geq 0$; as a result

$$\sum_{k=n}^{m-1} \varphi_\beta^k\big(d(\zeta_1,\zeta_0)\big) \longrightarrow 0 \quad \text{as } n,m \longrightarrow +\infty.$$

Therefore, ζ_n is a Cauchy sequence. Since (X,d) is complete and A is closed, then the sequence $\{\zeta_n\}$ converges to some element $\zeta \in A$.

On the other hand, we have

$$\begin{aligned}
d(\zeta, B) &\leq d(\zeta, T\zeta_n)\\
&\leq d(\zeta, \zeta_{n+1}) + d(\zeta_{n+1}, T\zeta_n)
\end{aligned}$$

$$= d(\zeta, \zeta_{n+1}) + d(A, B)$$
$$\leq d(\zeta, \zeta_{n+1}) + d(\zeta, B).$$

As $n \to \infty$, we get that the sequence $d(\zeta, T\zeta_n)$ converges to $d(\zeta, B)$. Using hypothesis (2) of the theorem, there exists a subsequence $\{\zeta_{n(k)}\}$ of $\{\zeta_n\}$ such that $T(\zeta_{n(k)})$ converges to some $\eta \in B$. Hence

$$d(A, B) \leq d(\zeta, \eta)$$
$$\leq d(\zeta, \zeta_{n_k+1}) + d(\zeta_{n_k+1}, T\zeta_{n_k}) + d(T\zeta_{n_k}, \eta)$$
$$= d(\zeta, \zeta_{n_k+1}) + d(A, B) + d(T\zeta_{n_k}, \eta).$$

As $n \to \infty$, we get that $d(\zeta, \eta) = d(A, B)$ and therefore $\zeta \in A_0$. Since $T(A_0) \subset B_0$, there exists $u \in A$ such that $d(u, T\zeta) = d(A, B)$. Consequently, we obtain $d(u, T\zeta) = d(\zeta_{n+1}, T\zeta_n) = d(A, B)$. Using the P-property, we deduce that $d(u, \zeta_{n+1}) = d(T\zeta, T\zeta_n)$.

Since T is special generalized β-proximal quasi-contractive, we obtain

$$d(u, \zeta_{n+1}) = d(T\zeta, T\zeta_n)$$
$$\leq \varphi\big(M_T(\zeta, \zeta_n)\big), \quad \forall n \in \mathbb{N}, \tag{3.8}$$

where

$$M_T(\zeta, \zeta_n) = \max\big\{\alpha_0 d(\zeta, \zeta_n), \alpha_1\big(d(\zeta, T\zeta) - d(A, B)\big),$$
$$\alpha_2\big(d(\zeta_n, T\zeta_n) - d(A, B)\big), \alpha_3\big(d(\zeta_n, T\zeta) - d(A, B)\big),$$
$$\alpha_4\big(d(\zeta, T\zeta_n) - d(A, B)\big)\big\}. \tag{3.9}$$

On the other hand, using the triangular inequality and (3.2), we have

$$M_T(\zeta, \zeta_n) \leq \max\big\{\alpha_0 d(\zeta, \zeta_n), \alpha_1\big(d(\zeta, T\zeta) - d(A, B)\big),$$
$$\alpha_2 d(\zeta_n, \zeta_{n+1}), \alpha_3\big(d(\zeta_n, \zeta) + d(\zeta, T\zeta) - d(A, B)\big),$$
$$\alpha_4 d(\zeta, \zeta_{n+1})\big\}. \tag{3.10}$$

Moreover, using the triangular inequality, we get

$$d(\zeta, T\zeta) \leq d(\zeta, \zeta_{n+1}) + d(\zeta_{n+1}, T\zeta_n) + d(T\zeta_n, T\zeta)$$
$$= d(\zeta, \zeta_{n+1}) + d(A, B) + d(T\zeta_n, T\zeta), \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Using inequality (3.11) and (3.2), we obtain that

$$d(\zeta, T\zeta) - d(\zeta, \zeta_{n+1}) - d(A, B)$$
$$\leq \varphi\big(M_T(\zeta_n, \zeta)\big), \quad \forall n \in \mathbb{N}. \tag{3.12}$$

Let $s = d(\zeta, T\zeta) - d(A, B)$, letting $n \longrightarrow +\infty$ in inequality (3.10), we get

$$\lim_{n \longrightarrow +\infty} M_T(\zeta_n, \zeta) \leq \max\{\alpha_1, \alpha_3\}s. \tag{3.13}$$

Suppose that $s > 0$. If φ is continuous, letting $n \longrightarrow +\infty$ in inequality (3.12) and using the fact that φ is nondecreasing, we get

$$s \leq \varphi(\max\{\alpha_1, \alpha_3\}s) \leq \varphi(\beta s) = \varphi_\beta(s) < s,$$

which is a contradiction. If $\beta > \max\{\alpha_1, \alpha_3\}$. We claim that also $s = 0$. Suppose that $s > 0$. Using inequality (3.13) and the definition of the limit, there exist $\varepsilon > 0$ and $N > 0$ such that, for all $n > N$, we have $M_T(\zeta_n, \zeta) < (\max\{\alpha_1, \alpha_3\} + \varepsilon)s$. Since φ is nondecreasing, from (3.13), we get

$$\begin{aligned}
d(\zeta, T\zeta) - d(\zeta, \zeta_{n+1}) - d(A, B) &\leq \varphi(M_T(\zeta_n, \zeta)) \\
&\leq \varphi((\max\{\alpha_1, \alpha_3\} + \varepsilon)s) \\
&= \varphi_\beta\left(\frac{\max\{\alpha_1, \alpha_3\} + \varepsilon}{\beta}s\right) \\
&< \frac{\max\{\alpha_1, \alpha_3\} + \varepsilon}{\beta}s < s. \tag{3.14}
\end{aligned}$$

By letting $n \longrightarrow +\infty$ in (3.14), we get

$$s < \frac{\max\{\alpha_1, \alpha_3\} + \varepsilon}{\beta}s < s,$$

which is a contradiction. Therefore $s = 0$ and so $d(\zeta, T\zeta) - d(A, B) = 0$, which implies that $d(\zeta, T\zeta) = d(A, B)$, that is, ζ is the best proximity point.

For the uniqueness, suppose that there are two distinct best proximity points ζ and η such that $d(\zeta, T\zeta) = d(\eta, T\eta) = d(A, B)$. Using the P-property, we get $d(\zeta, \eta) = d(T\zeta, T\eta)$. On the other hand, since T is special generalized proximal β-quasi contractive, we deduce that

$$d(\zeta, \eta) = d(T\zeta, T\eta) \leq \varphi(M_T(\zeta, \eta)),$$

where

$$M_T(\zeta, \eta) = \max\{\alpha_0 d(\zeta, \eta), \alpha_3(d(\eta, T\zeta) - d(A, B)), \alpha_4(d(\zeta, T\eta) - d(A, B))\}.$$

Using triangular inequalities in $M_T(\zeta, \eta)$ and the fact that ζ and η are best proximity points of T, we get

$$M_T(\zeta, \eta) \leq \{\alpha_0, \alpha_3, \alpha_4\}d(\zeta, \eta).$$

Let $r = d(\zeta, \eta)$. Since φ is nondecreasing, we obtain

$$r \leq \varphi(\{\alpha_0, \alpha_3, \alpha_4\}r) \leq \varphi(\beta r) = \varphi_\beta(r) < r,$$

which is a contradiction. □

Example 3.3 Consider the complete metric space $X = \mathbb{R}$ with the metric $d(\zeta, \eta) = |\zeta - \eta|$. Let $A = [0, 2]$ and $B = [4, 6]$. These sets are closed on the metric space (X, d). Also, since B is compact, then $B = [4, 6]$ is approximately compact with respect to $A = [0, 2]$. Also, let $T : A \longrightarrow B$ be defined by $T(\zeta) = 6 - \zeta$. Then it is easy to see that $d(A, B) = 2$ and $A_0 = \{2\}$, $B_0 = \{4\}$. Thus, $T(A_0) = T(\{2\}) = \{4\} = B_0$. Now we shall show that T is special generalized proximal β-quasi contractive with $\phi(t) = \frac{1}{2}t$, $\beta = 2$, and $\alpha_1 = 2$ and $\alpha_i = 0$ for $i = 1, 2, 3, 4$. Note that $\phi(t) = \frac{1}{2}t \in \Phi_2 \subset \Phi_1$ using Remark 2.3.

$$0 = d(T\zeta, T\eta) = |\zeta - \eta| \leq \frac{1}{2}\max\{2d(\zeta, \eta), 0, 0, 0, 0\}.$$

The function $\phi(t)$ is continuous mappings as well as $\beta = 2 > \max\{\alpha_1, \alpha_3\} = 0$. We deduce, using our Theorem 3.2, that T has a unique best proximity point which is $\zeta_* = 2$ in this example.

$$d(\zeta_*, T\zeta_*) = d\big(2, T(2)\big) = d(2, 4) = d(A, B) = 2.$$

4 Consequences

Several consequences of the main results of Sect. 3 are established next.

First, as an application to best proximity points, we propose the following results, which are an immediate consequence of our main Theorem 3.2.

Corollary 4.1 *Let* (A, B) *be a pair of nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Consider a non-self-mapping* $T : A \longrightarrow B$ *satisfying the following conditions*:

(1) $T(A_0) \subset B_0$ *and the pair* (A, B) *satisfies the P-property.*

(2) B *is approximately compact with respect to* A.

(3) *There exist elements* $\zeta_0, \zeta_1 \in A$ *such that*

$$d(\zeta_1, T\zeta_0) = d(A, B).$$

(4) *There exists* $\varphi \in \Phi_2$ *such that*

$$d(T\zeta, T\eta) \leq \varphi\big(M(\zeta, \eta)\big), \quad \forall \zeta, \eta \in A, \tag{4.1}$$

where

$$M(\zeta, \eta) = \max\bigg\{d(\zeta, \eta), d(\zeta, T\zeta) - d(A, B)),$$
$$d(\eta, T\eta) - d(A, B), \frac{d(\eta, T\zeta) + d(\zeta, T\eta)}{2} - d(A, B)\bigg\}.$$

Then T *has the unique best proximity point* $\zeta \in A$ *such that* $d(\zeta, T\zeta) = d(A, B)$.

Proof Note that the above quantity $M(\zeta, \eta)$ was introduced by Jleli, Karapinar, and Samet in [13].

The main idea is that

$$M(\zeta, \eta) \leq M_T(\zeta, \eta),$$

where

$$M_T(\zeta,\eta) = \max\big\{d(\zeta,\eta), d(\zeta,T\zeta) - d(A,B)),$$
$$d(\eta,T\eta) - d(A,B), d(\eta,T\zeta) - d(A,B), d(\zeta,T\eta) - d(A,B)\big\}.$$

So $\alpha_i = 1$ for all $i \in \{0,1,\ldots,4\}$. So for $\beta \geq 2 > \max\{\alpha_1,\alpha_3\} = 1$. According to our Theorem 3.2, if the comparison function $\varphi \in \Phi_2$, then T has a unique proximity point in A. $\square$

Corollary 4.2 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Consider a non-self-mapping $T : A \longrightarrow B$ satisfying the following conditions*:

(1) *$T(A_0) \subset B_0$ and the pair (A,B) satisfies the P-property.*

(2) *B is approximately compact with respect to A.*

(3) *There exist elements $\zeta_0, \zeta_1 \in A$ such that*

$$d(\zeta_1, T\zeta_0) = d(A,B).$$

(4) *There exists $q \in [0,1)$ such that*

$$d(T\zeta, T\eta) \leq qM(\zeta,\eta), \quad \forall \zeta,\eta \in A,$$

where

$$M(\zeta,\eta) = \max\big\{d(\zeta,\eta), d(\zeta,T\zeta) - d(A,B),$$
$$d(\eta,T\eta) - d(A,B), d(\eta,T\zeta) - d(A,B), d(\zeta,T\eta) - d(A,B)\big\}.$$

Then T has a unique best proximity point $\zeta \in A$ such that $d(\zeta,T\zeta) = d(A,B)$.

Proof Let $\varphi = qt$, which belongs to Φ_1 and is continuous. According our Theorem 3.2, T has a unique proximity point in A. $\square$

Before proposing consequences of our result to the existence and uniqueness of fixed points for self-mappings, we introduce the following definition.

Definition 4.3 Let A be a nonempty set of a metric space (X,d). A self-mapping $T : A \to A$ is called generalized β-quasi contractive if there exists a function $\varphi \in \Phi_\beta$, where $\beta > 0$ such that, for all $\zeta,\eta \in A$, we have

$$d(T\zeta, T\eta) \leq \varphi\big(M_T(\zeta,\eta)\big),$$

where

$$M_T(\zeta,\eta) = \max\big\{\alpha_0 d(\zeta,\eta), \alpha_1 d(\zeta,T\zeta), \alpha_2 d(\eta,T\eta), \alpha_3 d(\zeta,T\eta), \alpha_4 d(\eta,T\zeta)\big\},$$

with $\alpha_k \geq 0$ for $k = 0,\ldots,4$.

Several papers dealt with fixed point theory in the context of the generalizing of Banach's principle as in [14–20]. By our generalized β-quasi contractive mapping, we can propose some theorems on the existence and uniqueness of fixed points in complete spaces in a simple way.

Corollary 4.4 *Let (X, d) be a nonempty complete metric space. Consider a self-mapping $T : X \longrightarrow X$. Suppose that there exists $\beta \geq \max_{0 \leq k \leq 3}\{\alpha_k, 2\alpha_4\}$ such that T is a β-quasi contractive mapping.*

Moreover, assume that one of the following conditions holds:
- *φ is continuous*;
- *$\beta > \max\{\alpha_1, \alpha_3\}$.*

Then T has a unique fixed point in X.

Proof This is an immediate consequence of our main Theorem 3.2 since $A = B = X$ and every set is approximately compact with its self. Moreover, the notion of special generalized β-proximal quasi-contractive on the self-mapping case is exactly a β-quasi-contractive one.   □

Also the famous Cirić theorem is an immediate consequence of our theorem.

Corollary 4.5 *Let (X, d) be a complete metric space, and let $T : X \longrightarrow X$ be a quasi-contraction, that is,*

$$d(T\zeta, T\eta) \leq q \max\big\{d(\zeta, \eta), d(\zeta, T\zeta), d(\eta, T\eta), d(\zeta, T\eta), d(\eta, T\zeta)\big\}$$
$$\textit{for all } \zeta, \eta \in X, \tag{4.2}$$

where $q \in [0, 1)$ is some constant. Then T has a unique fixed point $\zeta \in X$.

Proof Using our main Theorem 3.2, since $A = B = X$ and every set is approximately compact with its self, the function $\varphi(t) = qt$, which is continuous and belongs to the set Φ_1. □

5 Conclusion

Improvements to some best proximity point theorems are proposed. This has been achieved by introducing a suitable mapping called special generalized proximal β-quasi contractive. These are non-self-mappings involving β-comparison functions. As an application, we establish the existence and uniqueness of well-known fixed point results for the case of self-mappings on complete metric spaces. We confirm our result by a suitable example.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the preparation of the paper. The authors read and approved the final manuscript.

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