Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2117–2125
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A new numerical method for heat equation subject to integral specifications

H. M. Jaradat[a,b], M. M. M. Jaradat[c,*], Fadi Awawdeh[d,b], Zead Mustafa[c], O. Alsayyed[d]

[a] *Department of Mathematics, Al al-Bayt University, Jordan.*
[b] *Department of Mathematics and Applied Sciences, Dhofar University, Salalah, Oman.*
[c] *Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar.*
[d] *Department of Mathematics, Hashemite University, Jordan.*

Communicated by W. Shatanawi

Abstract

We develop a numerical technique for solving the one-dimensional heat equation that combine classical and integral boundary conditions. The combined Laplace transform, high-precision quadrature schemes, and Stehfest inversion algorithm are proposed for numerical solving of the problem. A Laplace transform method is introduced for solving considered equation, definite integrals are approximated by high-precision quadrature schemes. To invert the equation numerically back into the time domain, we apply the Stehfest inversion algorithm. The accuracy and computational efficiency of the proposed method are verified by numerical examples.

Keywords: Heat equation, nonlocal boundary value problems, Laplace inversion, high-precision quadrature schemes, Stehfest inversion algorithm.
2010 MSC: 35K05, 34B10.

1. Introduction

In 1963, a nonlocal boundary equation was presented by Cannon [5], and Batten [3], independently. Then, parabolic initial-boundary problems with nonlocal integral conditions for parabolic equations were investigated by Kamynin [24] and Ionkin [23]. Over the recent years nonclassical problems for partial differential equations have been widely used for descriptions of a number of phenomena in modern physics

*Corresponding author
Email addresses: `husseinjaradat@yahoo.com` (H. M. Jaradat), `mmjst4@qu.edu.qa` (M. M. M. Jaradat), `fawawadeh@du.edu.om` (Fadi Awawdeh), `zead@qu.edu.qa` (Zead Mustafa), `o.alsayyed@yahoo.com` (O. Alsayyed)

Received 2015-12-04

and technology. Nonclassical problems with nonlocal conditions include relations between boundary values of an unknown solution and its derivatives and their values at internal points of a domain. Nonlocal problems with integral conditions which are natural generalizations of discrete nonlocal conditions, can be considered as mathematical models of processes with inaccessible boundaries.

This paper is focused on the numerical solution of the following diffusion equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = q(x,t), \qquad x \in (0,1), \quad 0 < t < T, \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x), \qquad x \in (0,1), \quad 0 < t \le T, \tag{1.2}$$

and the integral conditions

$$\int_0^1 u(x,t)dx = g_1(t), \qquad 0 < t < T, \tag{1.3}$$

$$\int_0^1 b(x)u(x,t)dx = g_2(t), \qquad 0 < t < T, \tag{1.4}$$

where $q(x,t),\ f(x), g_1(t), g_2(t), b(x)$ are known functions and T is a given constant. The mathematical modeling of this type of problems is encountered in heat transmission theory, in thermoelasticity, and in plasma physics [32, 33] and can be reduced to the nonlocal problems. Therefore, partial differential equations with nonlocal boundary conditions received much attention in the last 20 years. However, most of the papers were devoted to the second-order parabolic equations, particularly to heat conduction equations. Recently parabolic equations with nonlocal boundary conditions have been treated extensively by finite difference methods, finite element procedures, boundary element techniques, spectral schemes, Adomian decomposition method, and the semi-discretization procedures [12, 14, 15, 18]. The numerical techniques developed in [16] are based on three-level explicit finite difference procedures. Ang [1] developed a numerical technique for solving the studied model. In [30], a different approach is used by using combined finite-difference and spectral methods for solving the hyperbolic equations with integral condition. The proof of the existence, uniqueness and continuous dependence of the strong solution upon the data for an initial-boundary value problem and integral conditions for this problem is studied by Bouziani [4]. A theoretical discussion of these cases of equations can be found in [6, 7]. The famous work of Lin [26] was one of the first papers devoted to solving similar parabolic inverse problems. Dehghan applied some finite-difference schemes [13, 17] and a shifted Tau method [19] for solving similar problem. Cases of similar forms of those parabolic equations have been considered by various authors [8, 9, 21].

The purpose of the present article is to give a method of solving problem (1.1) under initial condition (1.2) and integral conditions (1.3) and (1.4) by using the Laplace transform technique. Often the analytical inverse transform is too difficult to find or evaluate in closed form. Numerical inversion methods are then used to overcome this difficulty. There are many approximate Laplace inversion algorithms. In this paper we will use the Stehfest inversion algorithm [34] in order to invert the Laplace transform. We use the high-precision quadrature schemes for numerical approximation of the integrals appearing in the method.

The rest of the paper is organized as follows. In the next section, we introduce the high-precision quadrature schemes which can be used to approximate definite integrals. The Laplace transform technique to solve problem (1.1)–(1.4) is used in Section 2. Moreover, we get an analytical solution representation by using the Stehfest inversion algorithm. Numerical results are given in Section 3. Finally, Section 4 contains our conclusions of the paper.

2. The Quadrature Scheme

In order to introduce the used method, some preliminary explanations are needed. Let a finite interval (a,b) be given, as well as positive integer numbers m and n. We set $h = (b-a)/n$ and $h_j = a + jh$ (and

B_{2i} for Bernoulli numbers). From Euler–Maclaurin formula, assuming that the function has at least $2m+2$ continuous derivatives, it follows

$$I = \int_a^b f(x)dx = h\sum_{j=0}^{n} f(x_j) - \frac{h}{2}\left(f(a)+f(b)\right) - \sum_{i=1}^{m}\frac{h^{2i}B_{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right) - E,$$

where

$$E = \frac{h^{2m+2}(b-a)B_{2m+2}f^{(2m+2)}(\xi)}{(2m+2)!}$$

for a $\xi \in (a,b)$. For more details, one can refer to [2, 35, 36, 37].

Transforming the integral of $f(x)$ on the interval $[-1,1]$ to an integral on $(-\infty,\infty)$ can be done by using the substitution of variable $x = g(t)$. In this case, we can write, for $h > 0$,

$$I = \int_{-1}^{1} f(x)dx = \int_{-\infty}^{\infty} f(g(t))g'(t)dt = h\sum_{j=-\infty}^{\infty} w_j f(x_j) + E = I_h + E, \tag{2.1}$$

where $x_j = g(h_j)$ and $w_j = g'(h_j)$. We truncate the infinite summation (2.1) into a finite one as

$$I_h^{(N)} = h\sum_{j=-N^-}^{N^+} w_j f(x_j).$$

Here $N = N^- + N^+ + 1$ is the number of the sampling points actually used.

For the integral

$$I = \int_{-1}^{1} f(x)dx,$$

the transformation

$$x = g(t) = \tanh\left(\frac{\pi}{2}\sinh t\right)$$

gives the double exponential formula

$$I_h = \frac{\pi}{2}h\sum_{j=-\infty}^{\infty} f\left(\tanh\left(\frac{\pi}{2}\sinh hj\right)\right)\frac{\cosh hj}{\cosh^2\left(\frac{\pi}{2}\sinh hj\right)}.$$

Note that the abscissas x_j and the weights w_j can be computed only one time for a given h, and then used for other problems. Typically, one selects $h = 2^{-m}$, for some m. It is found that $m = 12$ is more than sufficient to evaluate most integrals to 500-digit accuracy. One typically proceeds one "level" at a time, where level k uses $h = 2^{-k}$, starting with level one and continuing until either a fully accurate result is obtained or the final (m-th) level is completed. For more details refer to [2, 35].

3. Analysis of the method

We divide this section into two subsections: the first one deals with the method based on the Laplace transform and the second one deals with Laplace inversion schemes.

3.1. Method based on Laplace transform

First we take the Laplace transform on both sides of (1.1)–(1.4) with respect to t, and get

$$\frac{d^2}{dx^2}U(x,s) - (s+1)U(x,s) = -f(x) - Q(x,s), \tag{3.1}$$

$$\int_0^1 U(x,s)dx = G_1(s), \tag{3.2}$$

$$\int_0^1 b(x)U(x,s)dx = G_2(s), \tag{3.3}$$

where $U(x,s) = \mathcal{L}[u(x,t)]$, $Q(x,s) = \mathcal{L}[q(x,t)]$, $G_1(s) = \mathcal{L}[g_1(t)]$ and $G_2(s) = \mathcal{L}[g_2(t)]$. We have now a boundary value problem governed by a second-order inhomogeneous ordinary differential equation. The general solution of (3.1) can be given by

$$U(x,s) = C_1(s)e^{\sqrt{s+1}x} + C_2(s)e^{-\sqrt{s+1}x} - \frac{1}{\sqrt{s+1}}\int_0^x [f(\tau)+Q(\tau,s)] \times \sinh(\sqrt{s+1}\,(x-\tau))d\tau. \tag{3.4}$$

Substituting (3.4) into the integral conditions (3.2)–(3.3), we have

$$\begin{aligned} C_1(s)\left[e^{\sqrt{s+1}}-1\right] - C_2(s)\left[e^{-\sqrt{s+1}}-1\right] =& \sqrt{s+1}G_1(s) + \frac{1}{\sqrt{s+1}}\int_0^1 [f(\tau)+Q(\tau,s)] \\ & \times [\cosh(\sqrt{s+1}\,(1-\tau))-1]d\tau. \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & C_1(s)\left[\int_0^1 b(x)e^{\sqrt{s+1}x}dx\right] + C_2(s)\left[\int_0^1 b(x)e^{-\sqrt{s+1}x}dx\right] \\ & = G_2(s) + \frac{1}{\sqrt{s+1}}\int_0^1 \left[[f(\tau)+Q(\tau,s)] \times \int_\tau^1 b(x)\sinh(\sqrt{s+1}\,(x-\tau))dx\right]d\tau. \end{aligned} \tag{3.6}$$

Solving (3.5)–(3.6) for $C_1(s)$ and $C_2(s)$, we have

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{3.7}$$

where

$$a_{11}(s) = e^{\sqrt{s+1}} - 1, \qquad a_{12}(s) = 1 - e^{-\sqrt{s+1}}, \tag{3.8}$$

$$\begin{aligned} b_1(s) = \sqrt{s+1}G_1(s) + \frac{1}{\sqrt{s+1}}\int_0^1 [f(\tau)+Q(\tau,s)] \\ \times [\cosh(\sqrt{s+1}\,(1-\tau))-1]d\tau, \end{aligned} \tag{3.9}$$

$$\tag{3.10}$$

$$a_{21}(s) = \int_0^1 b(x)e^{\sqrt{s+1}x}dx, \qquad a_{22}(s) = \int_0^1 b(x)e^{-\sqrt{s+1}x}dx, \tag{3.11}$$

$$\begin{aligned} b_2(s) = G_2(s) + \frac{1}{\sqrt{s+1}}\int_0^1 [f(\tau)+Q(\tau,s)] \\ \times \int_\tau^1 b(x)\sinh(\sqrt{s+1}\,(x-\tau))dx]d\tau. \end{aligned} \tag{3.12}$$

Thus, to find a solution in Laplace domain, one has to evaluate all the integrals appearing in (3.8)–(3.12). Using high-precision quadrature schemes, we have the following approximations of the above integrals

$$\begin{aligned} \int_0^1 K(x)e^{\pm\sqrt{s+1}x}dx &= \frac{1}{2}\int_{-1}^1 K\left(\frac{1}{2}(x+1)\right)e^{\pm\sqrt{s+1}\left(\frac{1}{2}(x+1)\right)}dx \\ &\simeq \frac{\pi}{2}h\sum_{j=-\infty}^{\infty} w_j K\left(\frac{1}{2}(x_j+1)\right)e^{\pm\frac{1}{2}\sqrt{s+1}(x_j+1)}, \end{aligned}$$

$$\begin{aligned}
&\int_0^1 F(\tau,s)[\cosh(\sqrt{s+1}\,(1-\tau))-1]d\tau \\
&\qquad = \frac{1}{2}\int_{-1}^{1} F(\frac{1}{2}(\tau+1),s)\times[\cosh(\sqrt{s+1}\left(1-\frac{1}{2}(\tau+1)\right))-1]d\tau \\
&\qquad = \frac{1}{2}\int_{-1}^{1} F(\frac{1}{2}(\tau+1),s)\times[\cosh(\sqrt{s+1}\left(\frac{1}{2}(1-\tau)\right))-1]d\tau \\
&\qquad \simeq \frac{\pi}{2}h\sum_{j=-\infty}^{\infty} w_j F\left(\frac{1}{2}(x_j+1),s\right) x(\cosh(\sqrt{s+1}\left(\frac{1}{2}(1-x_j)\right))-1)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left[F(\tau,s)\times\int_\tau^1 K(x)\sinh(\sqrt{s+1}\,(x-\tau))dx\right]d\tau \\
&\simeq \frac{\pi}{2}h\sum_{j=-\infty}^{\infty} w_j F\left(\frac{1}{2}(x_j+1),s\right)\times\left(\frac{1-\frac{1}{2}(x_j+1)}{2}\right) h\sum_{i=-\infty}^{\infty} w_i K\left(\frac{1-\frac{1}{2}(x_j+1)}{2}x_i+\frac{1+\frac{1}{2}(x_j+1)}{2}\right) \\
&\quad\times\sinh(\sqrt{s+1}\left(\frac{1-\frac{1}{2}(x_j+1)}{2}x_i+\frac{1+\frac{1}{2}(x_j+1)}{2}-\frac{1}{2}(x_j+1)\right),
\end{aligned}$$

where the nodes x_j and the weights w_j are given by

$$x_j=\tanh\left(\frac{\pi}{2}\sinh hj\right),\quad w_j=\frac{\cosh hj}{\cosh^2\left(\frac{\pi}{2}\sinh hj\right)}. \tag{3.13}$$

3.2. Laplace Inversion Schemes

The Gaver–Stehfest algorithm for numerical inversion of the Laplace transform was developed in the late 1960s. Due to its simplicity and good performance, its popularity increases in such diverse areas as geophysics, operations research, economics, financial and actuarial mathematics, computational physics, engineering and chemistry [20, 22].

Assume that $f:(0,\infty)\to R$ is a locally integrable function, such that its Laplace transform

$$\mathcal{L}[f(t)]=F(s)=\int_0^\infty f(t)e^{-st}dt,$$

is finite for all $s>0$. The problem consists in recovering the original function $f(t)$ given that we know $F(s)$. This problem has numerous applications, and it has attracted a lot of attention from researchers over the last fifty years (see [10] for an up-to-date exposition of this area). The exact inversion is normally difficult to carry out, so approximate inversion techniques are used. There are many approximate Laplace inversion algorithms. (For more details see [28, 29, 31, 38]).

The numerical inversion of the Laplace transform arises in many areas of science and engineering. Stehfest [34] derived the Gaver–Stehfest algorithm for a numerical inversion of Laplace transforms. For most of the more interesting problems, however, numerical inverting often has numerical accuracy problems [11, 25, 27]. As such, small rounding errors in a computation may significantly offset the results, rendering these algorithms impractical to apply.

The Gaver–Stehfest method uses the summation

$$f(t)=\frac{\ln 2}{t}\sum_{n=1}^{N}\alpha_n F\left(\frac{n\ln 2}{t}\right). \tag{3.14}$$

The γ_n coefficients only depend on the number of expansion terms, N (which must be even). They are

$$\alpha_n=(-1)^{n+N/2}\sum_{k=\lfloor(n+1)/2\rfloor}^{\min\{n,N/2\}}\frac{k^{N/2}\,(2k)!}{(N/2-k)!k!\,(k-1)!\,(n-k)!\,(2k-n)!}. \tag{3.15}$$

The α_n coefficients become very large and alternate in sign when n increases. The precision of the Stehfest inversion method depends on the Stehfest number N. Indeed, one can see in equation (3.14) that the inversion is based on summation of N weighted values. The default Stehfest number is often chosen in the range $6 \leq N \leq 18$.

Taking this into account, we can obtain a solution to problem (1.1)–(1.4) as

$$u(x,t) = \frac{\ln 2}{t}\sum_{n=1}^{N}\alpha_n[C_1 e^{\sqrt{\frac{n\ln 2}{t}+1}x} + C_2 e^{-\sqrt{\frac{n\ln 2}{t}+1}x}$$
$$- \frac{\sqrt{t}}{\sqrt{n\ln 2+t}}\int_0^x [f(\tau)+Q(\tau,\frac{n\ln 2}{t})]\times \sinh(\sqrt{\frac{n\ln 2}{t}+1}\,(x-\tau))d\tau],$$

where α_n is given by (3.15).

4. Applications

In this section, we illustrate efficiency and accuracy of the presented method by the following numerical examples.

Example 4.1. Consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = 2t + t^2 + x, \qquad x \in (0,1), \quad 0 < t < T,$$

with the initial condition

$$u(x,0) = x, \quad x \in (0,1), \quad 0 < t \leq T,$$

and the integral conditions

$$\int_0^1 u(x,t)dx = \frac{1}{2} + t^2, \qquad 0 < t < T,$$
$$\int_0^1 xu(x,t)dx = \frac{1}{3} + \frac{1}{2}t^2, \qquad 0 < t < T.$$

We can verify that the exact solution to this problem is $u(x,t) = x + t^2$. The absolute errors in the approximation are shown in Figure 1.

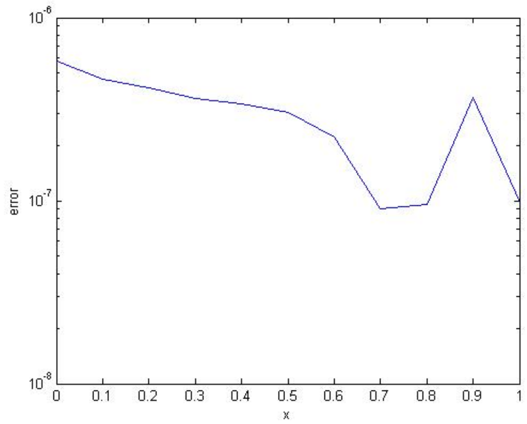

Figure 1: Absolute errors between the numerical and exact solutions in Example 4.1 for $N = 16$, $t = 1, x \in [0,1]$.

Example 4.2. Consider the heat equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u=11+6t+11t^2-x-2xt-xt^2-4x^2-8tx^2-4x^2t^2, \qquad x\in(0,1),\quad 0<t<T,$$

with the initial condition

$$u(x,0)=3-x-4x^2,\quad x\in(0,1),\quad 0<t\le T,$$

and the integral conditions

$$\int_0^1 u(x,t)dx=\frac{7}{6}(1+t^2), \qquad 0<t<T,$$

$$\int_0^1 (1+2x)u(x,t)dx=\frac{3}{2}+\frac{3}{2}t^2, \qquad 0<t<T.$$

The exact solution to this problem is $u(x,t)=\big(1+t^2\big)(3-x-4x^2\big)$. The absolute errors in the approximation are shown in Figure 2.

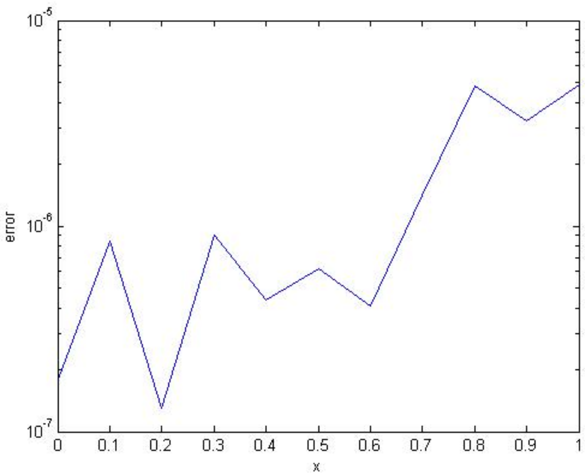

Figure 2: Absolute errors between numerical and exact solution in Example 4.2 for $N=16$, $t=2$, $x\in[0,1]$.

Example 4.3. Consider the heat equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+u=(10-2x)e^t, \qquad x\in(0,1),\quad 0<t<T,$$

with the initial condition

$$u(x,0)=5-x,\quad x\in(0,1),\quad 0<t\le T,$$

and the integral conditions

$$\int_0^1 u(x,t)dx=\frac{9}{2}e^t, \qquad 0<t<T,$$

$$\int_0^1 xu(x,t)dx=\frac{13}{6}e^t, \qquad 0<t<T.$$

The exact solution to this problem is $u(x,t)=(5-x)e^t$. The absolute errors in the approximation are shown in Figure 3.

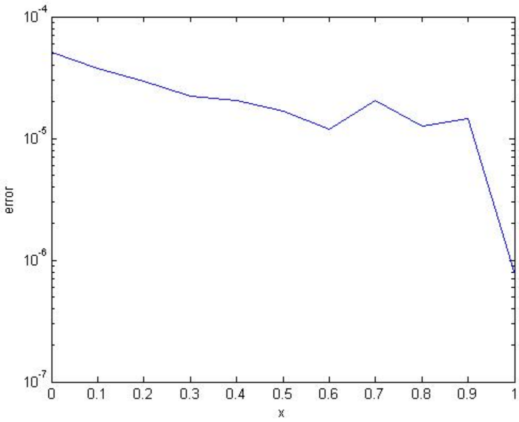

Figure 3: Absolute errors between numerical and exact solution in Example 4.3 for $N = 16$, $t = 0.4$, $x \in [0, 1]$.

5. Conclusions

We presented a computational method for solving the parabolic heat equation with an integral condition. A Laplace transform method is introduced for solving considered equation. Then high-precision quadrature schemes are used to approximate the resulting definite integrals. The Stehfest inversion algorithm is applied to invert the equation numerically, back into the time domain. The numerical results show that our new technique, described in this paper, is an accurate and reliable analytical technique, that worked very well for the studied problem. The new technique can be extended to high dimensional parabolic equations with integral conditions.

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