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On the solution of Bicriterion Inter Nonlinear fractional programs with fuzzy parameters in the objective functions

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في حل مشاكل البرمجة اللاخطية الصحيحة الكسرية ثنائية المعيار بوسائط فزية في دوال الهدف

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يتناول هذا البحث صياغة مشاكل البرمجة اللاخطية الصحيحة الكسرية ثنائية المعيار بوسائط فزية في دوال الهدف، كما عرضنا توصيف خوارزم حل للأنموذج محل الدراسة وقد تم تدعيم النتائج المستخلصة بمثال عددي توضيحي.

Key Words: Bicriterion, Integer Linear Fractional Programming, Fuzzy parameters, Fuzzy numbers, α-Optimality

ABSTRACT

In this paper we consider bicriterion integer nonlinear fractional programs having fuzzy parameters in the objective functions. For such programs, a solution algorithm is described to solve the formulated model. The results obtained in this paper have been illustrated by a numerical example.

INTRODUCTION

Some results in the field of integer nonlinear fractional programming can be found in [4,5]. In the presented paper , the authors give a solution procedure for solving bicriterion integer nonlinear fractional programs with fuzzy parameters in the objective functions. These fuzzy parameters are characterized by fuzzy numbers and the concept of (-optimality is introduced. The paper is organized as follows : In section 2, the problem formulation is considered and the concept of (-optimality is introduced together with the definition of (-level set of fuzzy numbers. In section 3, we propose an algorithm for solving the formulated model under consideration. In section 4,an illustrative simple example is given to clarify the developed theory. Finally, section 5 contains the conclusion.

PROBLEM FORMULATION

Let for i = 1,2 be vectors in (i =1, 2) are scalars and is a fuzzy parameter. Then, the bicriterion integer nonlinear fractional program with fuzzy parameter in the objective functions can be formulated as:

(BINLFP)$_{\tilde{\theta}}$

max z (x, θ) = [z_1(x,)$\tilde{\theta}$, z_2(x,)] $\tilde{\theta}$ (1a)

subject to

$$x \in M,$$

where the feasible region M is defined as:

$$M = \{ x \in R^n \mid Ax \leq b, x \geq 0 \textit{ and integer}\}, \quad (1b)$$

and

$$z_i(x, \tilde{\theta}) = \frac{(c_i^T + \tilde{\theta} h_i^T) x + v_i}{d_i^T x + \mu_i} \qquad (i = 1,2) \quad (1c)$$

are nonlinear fractional functions defined on M with fuzzy parameter ,$\tilde{\theta}$ A is an m x n matrix, $b \in R^n$ *and* R^n and is the set of all ordered n_tuples of real numbers. We assume that the feasible region M is a compact polyhedral set and that no where in M do any of the denominators (i = 1,2), take on a value of zero.

DEFINITION 1:

Let $\theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ be four real numbers , a real fuzzy number $\tilde{\theta}$ is a convex continuous fuzzy subset of the real line with a membership function $\mu\tilde{\theta}(\theta)$ which possesses the following properties:

(1) A continuous mapping from R^1 to the closed interval [0 , 1],

(2) $\mu_{\tilde{\theta}}(\theta) = 0$ *for all* $\theta \in [-\infty, \theta_1]$,

(3) *Strictly increa* sin *g on* $[q1, q2]$,

(4) $\mu_{\tilde{\theta}}(\theta) = 1$ *for all* $\theta \in [\theta_2, \theta_3]$,

(5) *Strictly decrea* sin *g on* $[\theta_3, \theta_4]$,

(6) $\mu_{\tilde{\theta}}(\theta) = 0$ *for all* $\theta \in [\theta_1, +\infty]$,

A possible shape of the fuzzy number $\mu_{\tilde{\theta}}$ *is illusrated in Fig.* 1.

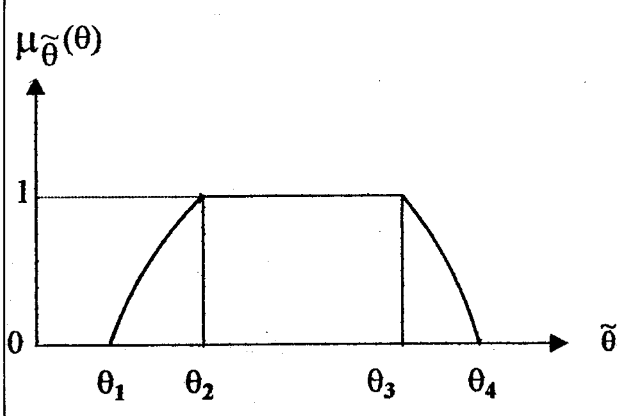

Fig. 1: Membership function of fuzzy number $\tilde{\theta}$

Now we assume that $\tilde{\theta}$ in problem (BINLFP)$\tilde{\theta}$(1a - 1b) is a fuzzy number whose membership function is $\mu_{\tilde{\theta}}(\theta)$

In what follows, we give the definition of (α-level set or (α-cut of fuzzy number $\tilde{\theta}$

DEFINITION 2:

The (α-level set of the fuzzy number $\tilde{\theta}$ is defined as the

$$L_\alpha(\tilde{\theta}) = \{\theta \in R \mid \mu_{\tilde{\theta}}(\theta) \geq \alpha \}$$

ordinary set $L_\alpha(\tilde{\theta})$ for which the degree of their membership function exceeds the level

$$\theta_1 \leq \theta_2 \quad \textit{iff} \quad L_{\alpha 1}(\tilde{\theta}) \supset L_{\alpha 2}(\tilde{\theta})$$

It should be noted that the level sets have the following property:

For a certain degree (α, problem (BINLFP)$_{\tilde{\theta}}$ can be understood as the following nonfuzzy α-bicriterion integer nonlinear fractional problem:-

(α-BINLFP)$_{\theta}$:

$$\textit{Max } z(x,\theta) = [z_1(x,\theta), z_2(x,\theta)], \qquad (2\text{ a})$$

subject to

$$x \in M(x,\theta),$$

where :

$M(x,\theta) = \{(x,\theta) \in R^{n+1} | Ax \leq b, \mu_{\tilde{\theta}}(\theta) \geq \alpha, x \geq 0$ *and integer*$\}$ (2 b)

and

$$z_1(x,\tilde{\theta}) = \frac{s_i^T + \tilde{\theta}h_i^T) x + v_i}{d_i^T x + \mu_i} \quad (i = 1,2) \quad (2\ c)$$

Based on the definition 2 of α-level set of the fuzzy number $\tilde{\theta}$, we introduce the concept of α-Pareto optimal solution to the **(α-BINLFP)$_\theta$** (2a) – (2c) as follows.

DEFINITION 3:

A point $x^* \in M(x,\theta)$ is said to be an α-Pareto optimal solution of the $(\alpha\text{-BINLFP})_\theta$ (2a) – (2c), if and only if there exists no other $x \in M(x,\theta)$, $\theta \in L_\alpha(\tilde{\theta})$ such that $z_i(x,q) \geq z_i(x^*,\theta^*)$, $(i = 1,2)$ with strictly inequality holding for at least on i, where the corresponding value of parameter θ^* is called α-level optimal parameter.

Problem **(θ-BINLFP)$_\theta$** can be written as :-

$(\alpha\text{-BINLFP})_\theta$:

$$Max\ z(x,\theta) = [z_1(x,\theta), z_2(x,\theta)], \quad (3\ a)$$

subject to

$$x \in M(x,\theta),$$

where :

$M(x,\theta) = \{(x,\theta) \in R^{n+1} | Ax \leq b, \ell^{(0)} \leq \theta \leq L^{(0)},$ $x \geq \theta$ *and integer*$\}$, (3 b)

Note that the constraint $\mu(\tilde{\theta}) \geq \alpha$ has been replaced by the equivalent constraint $\ell^0 \leq \theta \leq L^{(0)}$ *where* $\ell^{(0)}, L^{(0)}$ are lower and upper bounds on the variable θ .

In what follows, we shall state an equivalent bicriterion linear fractional program associated with program **(α-BINLFP)$_\theta$** (2a) – (2b) with the help of cutting-plane technique[2,3]. This equivalent program can be written in the form :

(θ-BINLFP)$_\theta$:

$$Max\ z(x,\theta) = [z_1(x,\theta), z_2(x,\theta)], \quad (4\ a)$$

subject to

$$x \in M_R^{(S)}, \quad (4\ b)$$

$$\ell^{(0)} \leq \theta \leq L^{(0)}, \quad (4\ c)$$

where $$z_1(x,\theta) = \frac{c_1^T + \theta h_1^T) x + v_1}{d_1^T x + \mu_1},$$

$$z_2(x,\theta) = \frac{c_2^T + \theta h_2^T) x + v_2}{d_2^T x + \mu_2}$$

and $M_R^{(s)} = \{ x \in R^n | A^{(s)} x \leq b^{(s)}, x \geq 0\}$

In addition,

$$A^{(s)} = \begin{bmatrix} A \\ \vdots \\ a_1 \\ \vdots \\ a_s \end{bmatrix} \quad \text{and} \quad b^{(s)} = \begin{bmatrix} b \\ \vdots \\ b_1 \\ \vdots \\ b_s \end{bmatrix}$$

are the original constraint matrix A and right_hand side vector b respectively, with s-additional constraints each corresponding to an efficient Gomory's fractional cut in the form $a_i x \leq b_i$ [2,3]. By an efficient Gomory's fractional cut we mean that a cut that is not redundant. It should be noted that $M_R^{(S)}$ is obtained by dropping the integer requirement on the variables x_j j = 1 , 2 , ... N in the set of constraints M defined by (1b).

For a certain $\theta = \theta^0$, we define the following values as in [6]:

$$z_i^* = \max \{z_i(x,\theta^0) | x \in M_R^s, i = 1.2\} \quad (5)$$

$$z_{*1} = \max \{z_1(x,\theta^0) | x \in M_R^s, z_2(x,\theta^0) \geq z_2^* \quad (6)$$

$$z_{*2} = \max \{z_2(x,\theta^0) | x \in M_R^s, z_1(x,\theta^0) \geq z_1^* \quad (7)$$

clearly,

$$z_{*1} \leq z_1(x,\theta^0) \leq z_1^*$$

Next, for $\lambda = \lambda^0$ in the interval $[z_{*1}, z_1^*]$, the following nonlinear fractional program $P(\theta^0,\lambda^0)$ in the form :

P (θ^0,λ^0):

$$Max\quad z_2(x,\theta^0), \quad (8a)$$

subject

$$x \in M_R^{(S)}, \quad (8b)$$

$$z_1(x,\theta^0) \geq \lambda^0 \quad (8c)$$

has an optimal solution x^0 which is efficient for program $(\theta\text{-BINLFP})_\theta$ (2a) – (2c)

Program $P(\theta^0,\lambda^0)$ can be written as : (8a) – (8c)

P (θ^0,λ^0):

$$\max\quad z_2(x,\theta^0) = \frac{c_2^T + \theta h_2^T) x + v_2}{d_2^T x + \mu_2} \quad (9a)$$

subject to

$$x \in M_R^{(S)}, \quad (9b)$$

$$z_1(x,\theta^0) = \frac{c_1^T + \theta h_1^T) x + v_2}{d_1^T x + \mu_1} \geq \lambda^0 \quad (9c)$$

Problem (9a) – (9c) above is a mixed-integer nonlinear fractional programming problem which can be solved using Charnes-Cooper transformation method [1] together with the help of cutting plane technique [2, 3]. This leads to the mixed-integer solution for $(\theta\text{-BINLFP})_{\tilde{\theta}}$ (1a) – (1c).

SOLUTION ALGORITHM:

The algorithm to solve problem $(\text{BINLFP})_{\tilde{\theta}}$ (1a) – (1c) can be summerized in the following steps:-

Step (1): Choose points $\theta_1, \theta_2, \theta_3, \theta_4$ to elicit a membership function for the fuzzy $\tilde{\theta}$ number in problem $(\text{BINLFP})_{\tilde{\theta}}$ (1a)-(1c) satisfying assumptions (1) - (6) in definition 1.

Step (2): For a certain degree α, formulate problem $(\alpha\text{-BINLFP})_\theta$ (2a) – (2b).

Step (3): Use the transformation method [1] with the help of cutting-plane technique [2,3] to convert problem $(\alpha\text{-BINLFP})_\theta$ (2a) – (2b) to problem $(\alpha\text{-BINLFPP})_\theta$ (4a) – (4b).

Step (4): Find z_{*1}, Z^*_1 the lower and upper bounds for the first objective function as stated before in theory.

Step (5): Choose $\lambda = \lambda^0 \in [z_{*1}, z^*_1]$ then formulate problem P $(\lambda = \lambda^0)$ (9a) – (9c)

Step (6): Use Eureka package with the help of cutting-plane technique [2,3] to solve problem (9a) – (9c)

AN ILLUSTRATIVE EXAMPLE:

In this section we provide a simple example to clarify the proposed algorithm. The problem is the following bicriterion integer nonlinear fractional program with fuzzy parameter $\tilde{\theta}$ in the objective functions :

$(\text{BINLFP})_{\tilde{\theta}}$:

$$\max\ z(x,\tilde{\theta}) = [\, z_1(x,\tilde{\theta}),\ z_2(x,\tilde{\theta})]$$

subject to

$$x \in \mathbf{M},$$

where the feasible region M is defined as:

$M = \{x \in R^2 \mid 2x_1 \leq 7, 4x_2 \leq 9, x_1, x_2 \geq 0$ *and integer*

and $\quad z_1(x,\tilde{\theta}) = \dfrac{(1+\tilde{\theta})\, x_1 + 1}{x_2 + 1}$

$$z_2(x,\tilde{\theta}) = \frac{(1+2\tilde{\theta})\, x_2 + 1}{x_1 + 1}$$

Let the fuzzy parameter is characterized by the following fuzzy numbers :

$$\tilde{\theta} = (0, 1, 4, 10)$$

Assume that the membership function corresponding to the fuzzy number is in the form:-

$$\mu_{\tilde{\theta}}(0) = \begin{cases} 0 & \theta \leq \theta_1 \\ 1 - \left\{\dfrac{\theta - \theta_1}{\theta_1 - \theta_2}\right\}^2 & \theta_1 \leq \theta \leq \theta_2 \\ 1 & \theta_2 \leq \theta \leq \theta_3 \\ 1 - \left\{\dfrac{\theta - \theta_3}{\theta_4 - \theta_3}\right\}^2 & \theta_3 \leq \theta \leq \theta_4 \\ 0 & \theta \leq \theta_4 \end{cases}$$

Consider that (α-level set of the fuzzy number $\tilde{\theta}$ is given by :

$\mu_{\tilde{\theta}}(\theta) \geq 0.36$, *then we get* $-0.8 \leq \theta \leq 8.8$

The nonfuzzy (α-bicriterion integer nonlinear fractional program can be written as :

$(\alpha\text{-BINLFP})_\theta$

$$\max\ z(x,\theta) = [z_1(x,\theta), z_2(x,\theta)],$$

subject to

$$x \in M(x,\theta),$$

$$M(x,\theta) = \{x \in \boldsymbol{R}^2 \mid 2x_1 \leq 7,$$
$$4x_2 \leq 9,$$
$$-0.8 \leq \theta \leq 8.8, \quad x_1, x_2 \geq 0 \text{ and integer}\},$$

$$z_1(x,\theta) = \frac{(1+\theta)\, x_1 + 1}{x_2 + 1},$$

$$z_2(x,\theta) = \frac{(1+2\theta)\, x_2 + 1}{x_1 + 1},$$

Using Charnes Cooper transformation [1], and using each objective individually we have the following two problems :

$\mathbf{P_1}$:

$$\max\ z_1(y_1, \rho_1, \theta) = (1+\theta)\, y_1, \rho_1,$$

subject to

$$y_2 + \rho_1 = 1,$$
$$y_1 \leq 3.50\, \rho_1,$$
$$y_2 \leq 2.25\, \rho_1,$$

$$\theta \le 8.8,$$
$$-\theta \le 0.8,$$
$$y_1, y_2 \ge 0, \quad \rho > 0,$$
$$\text{and} \quad \frac{y_1}{\rho_1}, \frac{y_2}{\rho_1} \quad \text{are integer}$$

Using Eureka Package neglecting the integer requirement in problem P_1 above, we obtain optimal solution

$y_1^* = 3.50, y_2^* = 0, \rho_1^* = 1, \theta^* = 8.79, z_1^* = 35.26.$

Similarly,

P_2:

$$\max \quad z_2(y_1, \rho_2, \theta) = (1 + 2\theta) y_2 + \rho_2,$$
subject to
$$y_1 + \rho_2 = 1,$$
$$y_1 \le 3.5,$$
$$y_2 \le 2.25 \rho_2,$$
$$\theta \le 8.8,$$
$$-\theta \le 0.8,$$
$$y_1, y_2 \ge 0, \quad \rho_2 > 0,$$
$$\text{and} \quad \frac{y_1}{\rho_2}, \frac{y_2}{\rho_2} \quad \text{are integer}$$

Using Eureka Package neglecting the integer requirement in problem P_2 above, we obtain optimal solution $y_1^* = 0.0024, y_2^* = 2.2434, \rho_1^* = 0.997, \theta^* = 0.798,$ $z_2^* = 6.82.$

It should be noted that the optimal solution of bicriterion integer nonlinear fractional program $(\text{BINLFP})_{\tilde{\theta}}$ with fuzzy parameter $\tilde{\theta}$ in the objective functions is the same solution to the following problem P:

P:

$$\max \quad z_2(y_1, \rho_2, \theta) = (1 - 2\theta) y_2 + \rho_2,$$
subject to
$$y_1 + \rho_2 = 1,$$
$$y_1 \le 3.5,$$
$$y_2 \le 2.25 \rho_2,$$
$$\theta \le 8.8,$$
$$-\theta \le 0.8,$$
$$(1 + \theta)\, y_1 + 0.692, \quad \rho_2 \ge 0.308 y_2$$
$$\text{and} \quad \frac{y_1}{\rho_2}, \frac{y_2}{\rho_2} \quad \text{are integer}$$

Solving problem P with the help of Gomory's mixed fractional cut, we obtain $y_1^* + 0, y_2^* + 2, \rho_1^* = 1,$ $\theta^* = -0.798$. This yields $x^* = (x_1^*, x_2^*) = (0,2)$ which is the optimal integer solution to bicriterion integer nonlinear fractional program (BINLFP) $\tilde{\theta}$ with fuzzy parameter $\tilde{\theta}$ in the objective functions with $\alpha \ge 0.36$.

CONCLUSIONS

In this paper we have proposed a solution procedure for solving bicriterion fuzzy integer nonlinear fractional programming problems. An illustrative numerical simple example has been given to clarify the developed theory and the proposed algorithm .

However, there are some open points of research which should be explored and studied in the field of fuzzy integer nonlinear fractional optimization problems.

Some of these points are :

(i) A solution procedure is needed for treating fuzzy bicriterion and multiple objective integer linear and nonlinear fractional programs with fuzzy parameters in the constraints and in the objective functions.

(ii) A parametric study should be carried out on the α-level set of fuzzy parameters in fuzzy integer linear and nonlinear fractional optimization problems.

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