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Generation of Sets of Sequences Isomorphic to Walsh Sequences

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توليد مجموعات من المتتاليات الأيزومورفية لمتتاليات والش

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إن مجموعة متتاليات Walsh من الرتبة 2^k (حيث k صحيح موجب) تشكل زمرة جمعية مولدة بمتتاليات Rademacher من الرتبة k، وباستثناء المتتالية الصفرية فإن مجموعة متتاليات Walsh هذه تشكل مجموعة متعامدة. هذا البحث يسمح بتوليد 2^{k-2} مجموعة من المتتاليات. كل من هذه المجموعات مكافئة لمجموعة متتاليات Rademacher من الرتبة k ونفس العدد من المجموعات الايزومورفية لمتتاليات Walsh الموافقة.

Keywords: *Walsh sequences, Rademacher sequences, Orthogonal functions.*

ABSTRACT

Walsh sequences of the order 2^k, k positive integer, form an additive group generated by Rademacher sequences set of k-order. Except the zero sequence, Walsh sequences form an orthogonal set. Our present work allows us to generate 2^{k-2} sets of sequences. Each of these sets is equivalent to the Rademacher sequences set of k-order, and to that related to Walsh sequences.

1. Introduction

In 1923, J. L. Walsh [1] defined a system of orthogonal functions that is complete over the normalized interval $(0,1)$.

The method of specifying the Walsh functions of arbitrary order $N=2^k$, $k=1,2,\ldots$ had been a problem of considerable difficulty until the year 1970, when Byrnes and Swick [2] showed that Walsh functions could be obtained from Rademacher functions and from the solutions of certain differential equations. Byrnes and Swick considered the inherent symmetry properties of the Walsh functions. Walsh functions of order N are defined as a set of N time functions, denoted by $\{W_j(t), t \in (0,T), j = 0,1,\ldots,N-1\}$, such that

- $W_j(t)$ takes the values $\{+1,-1\}$ except at the jumps, where it takes the value zero.
- $W_j(0)=1$ for all j.
- $W_j(t)$ has precisely j sign changes (zero crossings) in the interval $(0,T)$.
- $\int_0^T W_j(t).W_K(t)dt = \begin{cases} 0, j \neq k \\ T, j = k \end{cases}$
- Each function $W_j(t)$ is either odd or even with respect to the midpoint of the interval.

A set of Walsh functions is ordered according to the number of zero crossings (sign changes) $\{W_0(t), W_1(t),\ldots,W_j(t),\ldots,W_{n-1}(t)\}$. This set forms a multiplicative group. Graphs of the Walsh functions of order 8 are given in Figure 1.

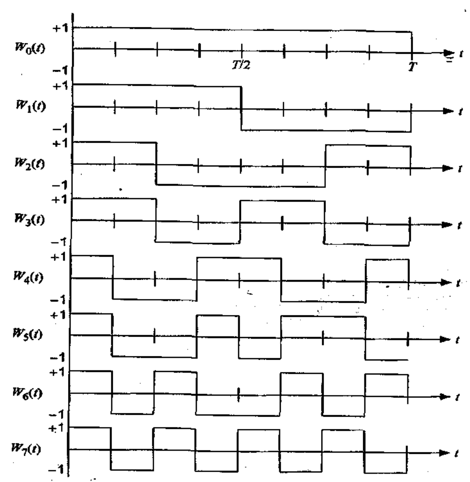

Figure 1. Walsh functions of order $8 = 2^3$

Walsh sequences of order 2^k, which are generated by the binary representation of Walsh functions of order $N = 2^k$, form a group under modulo 2 addition (addition group). The set of these sequences except W_0 forms an orthogonal set. Tables 1, 2 and 3 show the sequences of order 2^2, 2^3 and 2^4 respectively.

Index Sequence	Walsh Sequence of order $4=2^2$
0 0	W_0=0 0 0 0
0 1	W_1= 0 0 1 1
1 0	W_2= 0 1 1 0
1 1	W_3= 0 1 0 1

Table 1. Walsh sequences of order $4=2^2$

Index Sequence	Walsh Sequence of order $8=2^3$
0 0 0	W_0 =0 0 0 0 0 0 0 0
0 0 1	W_1 = 0 0 0 0 1 1 1 1
0 1 0	W_2 = 0 0 1 1 1 1 0 0
0 1 1	W_3 = 0 0 1 1 0 0 1 1
1 0 0	W_4 = 0 1 1 0 0 1 1 0
1 0 1	W_5 = 0 1 1 0 1 0 0 1
1 1 0	W_6 = 0 1 0 1 1 0 1 0
1 1 1	W_7 =0 1 0 1 0 1 0 1

Table 2. Walsh sequences of order $8=2^3$

Walsh Sequence																
W_0 =	(0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0)
W_1 =	(0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1)
W_2 =	(0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0)
W_3 =	(0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1)
W_4 =	(0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0)
W_5 =	(0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1)
W_6 =	(0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0)
W_7 =	(0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1)
W_8 =	(0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0)
W_9 =	(0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1)
W_{10} =	(0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0)
W_{11} =	(0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1)
W_{12} =	(0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0)
W_{13} =	(0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1)
W_{14} =	(0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0)
W_{15} =	(0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1)

Table 3. Walsh sequences of order $16=2^4$

The Walsh functions can be generated by any of the following methods:

1. Using Rademacher functions.
2. Using Hadamard matrices.
3. Exploiting the symmetry properties of Walsh functions [3,4].

Walsh functions (or sequences) are used widely as orthogonal sets in the forward and inverse link of communications channels in the CDMA systems especially in the pilot channels, the Sync channels and the Traffic channels [5,6].

This work aims to generate 2^k sets of sequences equivalent to Rademacher sequences of order *k*, and the same number of isomorphic sets of Walsh sequences set by using division ring and then to find recursive formulas to generate them.

2. Basic Definitions

Definition 1.

The set of functions $(f_i)_{i\in I}$ piecewise continuous on the interval $(0,T)$ is said to be orthogonal normalized and complete if the following conditions are satisfied:

1. Orthogonality: $\int_0^T f_i(t).f_j(t)\,dt = 0; i,j \in I, i \neq j$.
2. Normalization: $\int_0^T f_i^2(t)\,dt = T, \forall i \in I$.
3. Completion: each function piecewise continuous on the interval $(0,T)$ can be approximated by a linear combination of these functions [7].

Definition 2.

Rademacher functions $\{R_n(t)\ ;\ t\in(0,T), n=1,2,\ldots,\ \log_2 N = k\}$ of order k are a set of $1+\log_2 N$ orthogonal functions consisting of $N = 2^k$ rectangular pulses that assume alternately the values +1 and -1 in an interval of $(0,T)$.

The Rademacher $\{R_n(t)\}$ functions can also be defined by

$$R_n(t) = \text{sgn}(\sin 2^n \pi t),\ t \in (0,T),\ n = 1,2,\ldots,\log_2 N = k$$

where $R_0(t) \equiv 1$, and

$$\text{sgn}(x) \triangleq \begin{cases} -1\ ;\ for\ \ x<0 \\ \ \ 0\ ;\ for\ \ x=0 \\ \ \ 1\ ;\ for\ \ x>0 \end{cases}$$

The Rademacher functions are constructed as follows. First, $R_0(t)=1$, i.e, $R_0(t)$ is the function with the value 1 over the entire interval of duration T. Then, to obtain $R_1(t)$ divide the interval $(0,T)$ in half, and let the value of $R_1(t)$ be 1 in the first half and -1 in the second half.

$$R_1(t)=\begin{cases} 1, & t\in\left(0,\frac{T}{2}\right) \\ -1, & t\in\left(\frac{T}{2},T\right) \\ 0, & t=0,\frac{T}{2},T \end{cases}$$

Similarly, construct $R_2(t)$ in the form

$$R_2(t)=\begin{cases} 1, & t\in\left(0,\frac{T}{4}\right) and\ t\in\left(\frac{T}{2},\frac{3T}{4}\right) \\ -1, & t\in\left(\frac{T}{4},\frac{T}{2}\right) and\ t\in\left(\frac{3T}{4},T\right) \\ 0, & t=0,\frac{T}{4},\frac{T}{2},\frac{3T}{4},T \end{cases}$$

Recursively, each subinterval is divided into two halves. The values 1, -1 and 0 are given to the left half, right half and midpoint, respectively.

They are orthogonal and are represented in the form of binary sequences by using the logic representation $"1"\rightarrow 0$ and $"-1"\rightarrow 1$.

For example, the Rademacher sequences of order $k=3$ are

$$\begin{aligned} R_0 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ R_1 &= (0\ 0\ 0\ 0\ 1\ 1\ 1\ 1) \\ R_2 &= (0\ 0\ 1\ 1\ 0\ 0\ 1\ 1) \\ R_3 &= (0\ 1\ 0\ 1\ 0\ 1\ 0\ 1) \end{aligned}$$

The Rademacher sequences of order $k=4$ are

$$\begin{aligned} R_0 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ R_1 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1) \\ R_2 &= (0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1) \\ R_3 &= (0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1) \\ R_4 &= (0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1) \end{aligned}$$

Definition 3.

Suppose $x=(x_0,x_1,\ldots,x_{n-1})$ and $y=(y_0,y_1,\ldots,y_{n-1})$ are vectors on $GF(2)=\{0,1\}$ with length n. The autocorrelations function of x and y, denoted by $R_{x,y}$, is

$$R_{x,y}=\sum_{i=0}^{n-1}(-1)^{x_i+y_i}$$

where x_i+y_i is computed mod 2. It is equal to the number of agreements components minus the number of disagreements corresponding to components [2].

Definition 4.

Suppose G is a set of binary vectors of length n

$$G = \{X; X = (x_0, x_1, \dots, x_{n-1}), x_i \in F_2 = \{0,1\}, i = \{0, \dots, n-1\}$$

Let $1^* = -1$ and $0^* = 1$. The set G is said to be orthogonal if the following two conditions are satisfied

$$\forall X \in G, \quad \sum_{t=0}^{n-1} x^* \in \{-1,0,1\}, \quad \text{or} \quad |R_{x,0}| \le 1.$$

$$\forall X, Y \in G\ (X \ne Y), \quad \sum x_i^* y_i^* \in \{-1,0,1\}, \quad \text{or} \quad |R_{x,y}| \le 1.$$

That is, the absolute value of "the number of agreements minus the number of disagreements" is equal to or less than 1 [7].

3. Plan for Generating Walsh Sequences

3.1 First Step

We take the division ring $(Z/mZ,\ +\ ,\ \cdot\ )$, $m = 2^k$, with the mathematical operations $\bar{x} \cdot \bar{y} = \overline{x \cdot y}$ and $\bar{x} + \bar{y} = \overline{x + y}$, where $\bar{x}$ denotes left coset. Suppose

$$\psi : Z/mZ = \{\bar{0}, \bar{1}, \dots, \overline{2^K - 1}\} \to Z_{2^K} = \{0,1,2,\dots,2^K - 1\}, \quad \psi(\bar{x}) = x,$$

where ψ is an isomorphism and Z_{2^k} forms a ring for which the multiplicative operation is given in Table 4.

.	0	1	2	3	4	…	2^{k-1}	$2^{k-1}+1$	…	2^k-1
0	0	0	0	0	0	…	0	0	…	0
1	0	1	2	3	4	…	2^{k-1}	$2^{k-1}+1$	…	2^k-1
2	0	2	4	6	8	…	0	2	…	2^k-2
3	0	3	6	9	12	…	2^{k-1}	3	…	2^k-3
4	0	4	8	12	16	…	0	4	…	2^k-4
…	…	…	…	…	…	…	…	…	…	…
2^{k-1}	0	2^{k-1}	0	2^{k-1}	0	…	0	2^{k-1}	…	2^{k-1}
…	…	…	…	…	…	…	…	…	…	…
2^k-1	0	2^k-1	2^k-2	2^k-3	2^k-4	…	2^{k-1}	$2^{k-1}-1$	…	1

Table 4. Products of Z_{2^k}

Arranging up-down the rows $h_0, h_1, \dots, h_{2^k-1}$ and arranging left-right the columns $c_0, c_1, \dots, c_{2^k-1}$, we find

- All elements of h_0 are zero, and h_1 contains all elements of the ring in order.
- h_2 is double to h_1 and contains all multiples of 2 in order repeated twice or 2^1 times.

- $h_4 = h_2^2$ is double to h_2 and contains all multiples of 4 in order repeated 4 times or 2^2. Recursively, we find that the row h_{2^t}, $t = 1,2,\dots,k-1$, is obtained from the row $h_{2^{t-1}}$ by multiplying its products, and $h_{2^{k-1}}$ contains $(0\ 2^{k-1})$ repeated 2^{k-1} times but h_{2^k} is a zero row.

Suppose $\omega_1^{(1)} = [0 \quad 1]$ is a matrix whose elements are from Z_{2^k} and

$$\omega_k^{(1)} = \begin{bmatrix} h_{2^0} \\ h_{2^1} \\ \cdot \\ h_{2^{k-1}} \end{bmatrix}.$$

Then, $\omega_k^{(1)}$ can be obtained using the recursion

$$\omega_k^{(1)} = \begin{bmatrix} h_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 2\omega_{k-1}^{(1)} \ \vdots \ 2\omega_{k-1}^{(1)} \end{bmatrix}; k \geq 2.$$

Define the mapping

$$\chi : Z_{2^k} \to \{0,1\}; \chi\left(\{0,1,2,\dots\dots\dots,2^{k-1}-1\} \right) = 0 ; \chi\left(\{2^{k-1},\dots\dots\dots,2^k - 1\} \right) = 1$$

Suppose $\chi(H = \lfloor a_{ij} \rfloor_{n,m})$ (whose elements are of Z_{2^k}) $= \lfloor \chi(a_{ij}) \rfloor_{n,m}$ and $\chi(h_i) = H_i$. By computing the image of Table 4 along with χ, we get the binary representation of products defined on Z_{2^k}.

Suppose $(S)^K = \overbrace{(SS\dots S)}^{k\ time}$. Note that each double operation of h_1 halves every zero or one interval of binary representation into zero left half and one right half.

$$* H_0 = (0)^{2^K} \quad ; \quad H_1 = \left((0)^{2^{K-1}} (1)^{2^{K-1}}\right)$$

$$* H_2 = \left((0)^{2^{K-2}} (1)^{2^{K-2}}\right)^{2^1} \quad ; \quad H_4 = H_{2^2} = \left((0)^{2^{K-3}} (1)^{2^{K-3}}\right)^{2^2}$$

....

$$* H_{2^t} = \left((0)^{2^{K-t-1}} (1)^{2^{K-t-1}}\right)^{2^t}, \qquad t = 0,\dots,k-2$$

$$* H_{2^{K-1}} = (0 \quad 1)^{2^{K-1}} \quad ; \quad H_{2^K} = (0)^{2^K}$$

The rows $H_1, H_{2^1}, \dots, H_{2^{K-1}}$ are identical with Rademacher sequences $R_1, R_2, \dots, R_k$. These sequences are odd (the elements of the sequences symmetric about the midpoint are disagreements) and linearly independent. They can be obtained recursively from the matrix

$$W_K^{(1)} = \begin{bmatrix} H_1 \\ H_{2^1} \\ H_{2^2} \\ \vdots \\ H_{2^{K-1}} \end{bmatrix} = \begin{bmatrix} H_1 \\ \dots\dots\dots\dots\dots\dots \\ W_{K-1}^{(1)} \ \vdots \ W_{K-1}^{(1)} \end{bmatrix}, \quad K \geq 2,$$

where $W_1^{(1)} = [0 \quad 1]$ is called the seed matrix.

The matrix $W_K^{(1)}$ generates Walsh sequences of order 2^k, denoted by $\zeta_k^{(1)}$, by taking all linear combinations of the rows of $W_K^{(1)}$.

A study of Z_{2^k}; $k = 2,3,4,5,6$, shows that the binary representation of multiplication table on Z_{2^k} contains only the sequences

$$H_{2^{K-1}+2^{\lambda}} = H_{2^{K-1}} + H_{2^{\lambda}}, \quad \lambda = 0,\ldots,k-2 \qquad ,H_0 = R_0$$

i.e. the rows which are linear combinations of the rows of $W_K^{(1)}$. Consequently, the number of Walsh sequences in the table of binary representation is only *2k*.

For example, when $m = 2^4$, we have $Z_{2^4} = \{0,1,2,...,15\}$. The products on the ring Z_{2^4} and the binary representation of these products are given in Tables 5 and 6 respectively.

0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	2	4	6	8	10	12	14	0	2	4	6	8	10	12	14
3	0	3	6	9	12	15	2	5	8	11	14	1	4	7	10	13
4	0	4	8	12	0	4	8	12	0	4	8	12	0	4	8	12
5	0	5	10	15	4	9	14	3	8	13	2	7	12	1	6	11
6	0	6	12	2	8	14	4	10	0	6	12	2	8	14	4	10
7	0	7	14	5	12	3	10	1	8	15	6	13	4	11	2	9
8	0	8	0	8	0	8	0	8	0	8	0	8	0	8	0	8
9	0	9	2	11	4	13	6	15	8	1	10	3	12	5	14	7
10	0	10	4	14	8	2	12	6	0	10	4	14	8	2	12	6
11	0	11	6	1	12	7	2	13	8	3	14	9	4	15	10	5
12	0	12	8	4	0	12	8	4	0	12	8	4	0	12	8	4
13	0	13	10	7	4	1	14	11	8	5	2	15	12	9	6	3
14	0	14	12	10	8	6	4	2	0	14	12	10	8	6	4	2
15	0	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

Table 5. Products in Z_{2^4}

$$\omega_4^{(1)} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & \vdots & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 & \vdots & 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & \vdots & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \end{bmatrix}$$

.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
3	0	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1
4	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
5	0	0	1	1	0	1	1	0	1	1	0	0	1	0	0	1
6	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1
7	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1
8	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
9	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
10	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
11	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	0
12	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
13	0	1	1	0	0	0	1	1	1	0	0	1	1	1	0	0
14	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0
15	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0

Table 6. Binary representation of products in Z_{2^4}

$$W_4^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \vdots & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \vdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\xi_4^{(1)}$ corresponds to Table 3.

3.2 Second Step

Suppose $\varphi_\alpha : Z_{2^K} \to Z_{2^K}; \varphi_\alpha(X) = \alpha X; \alpha = 2\lambda - 1 \in Z_m; 1 < \alpha \leq 2^{K-1} - 1$ that is $\alpha = 2\lambda - 1$, $\lambda = 2,3,\ldots,2^{k-2}$. φ_α is a group isomorphism which preserves the outer multiplication operation such that

$$\varphi_\alpha(x+y) = \alpha(x+y) = \alpha\, x + \alpha\, y = \varphi_\alpha(x) + \varphi_\alpha(y)$$
$$\varphi_\alpha(ax) = \alpha(ax) = a(\alpha\, x) = a\varphi_\alpha(x) \quad ; \quad a \in Z_2 k$$
$$\varphi_\alpha^{-1} = \varphi_{\alpha^{-1}}$$

Suppose $\varphi_\alpha\left(\left[a_{ij}\right]_{n,m} \right) = \left[\varphi_\alpha\left(a_{ij}\right)\right]_{n,m}; a_{ij} \in Z_{2^k}$. Then $\varphi_\alpha\left(\omega_k^{(1)}\right) = \alpha\omega_k^{(1)}$.

We assume $\alpha\omega_k^{(1)} = \omega_k^{(\lambda)}$.

$$\varphi_\alpha\left(\omega_k^{(1)}\right) = \omega_k^{(\lambda)} = \begin{bmatrix} \alpha h_1 \\ \alpha h_{2^1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha h_{2^{k-1}} \end{bmatrix} = \begin{bmatrix} h_\alpha \\ h_{\alpha 2^1} \\ \cdot \\ \cdot \\ \cdot \\ h_{\alpha 2^{k-1}} \end{bmatrix}$$

where the indices are taken using mod 2^k.

$$\omega_k^{(\lambda)} = \begin{bmatrix} & h_\alpha & \\ \cdots & \cdots & \cdots \\ 2\alpha\omega_{k-1}^{(1)} & \vdots & 2\alpha\omega_{k-1}^{(1)} \end{bmatrix} = \begin{bmatrix} & h_\alpha & \\ \cdots & \cdots & \cdots \\ 2\alpha\omega_{k-1}^{(\lambda)} & \vdots & 2\alpha\omega_{k-1}^{(\lambda)} \end{bmatrix}$$

$$\chi\left(\omega_k^{(\lambda)}\right) = W_K^{(\lambda)} = \begin{bmatrix} H_\alpha \\ H_{\alpha 2^1} \\ \vdots \\ H_{\alpha 2^{K-1}} \end{bmatrix} = \begin{bmatrix} & H_\alpha & \\ \cdots & \cdots & \cdots \\ W_{k-1}^{(\lambda)} & \vdots & W_{k-1}^{(\lambda)} \end{bmatrix} \quad ,K \geq 2 \quad \lambda = 2,3,\ldots\ldots,2^{k-2}$$

where $W_1^{(\lambda)} = [0 \quad 1]$ is the seed matrix. Looking at Figure 2, and by symmetry, we see that the rows of $W_k^{(\lambda)}$ form an orthogonal set, which corresponds to the odd $W_k^{(1)}$.

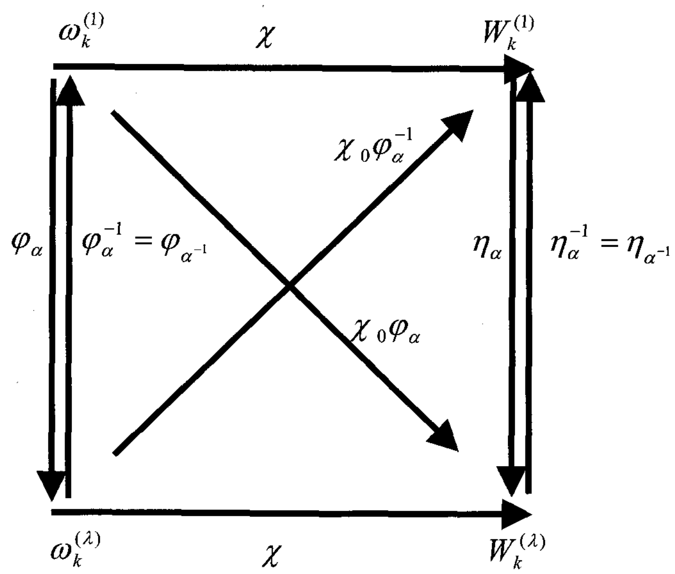

Figure 2. The correspondence between $W_k^{(1)}$ and $W_k^{(\lambda)}$

The matrix $W_k^{(\lambda)}$ generates a set of orthogonal sequences, denoted by $\xi_k^{(\lambda)}$. It is an isomorphism to $\xi_k^{(1)}$ and consists of all linear combinations of the rows of $W_k^{(\lambda)}$.

The binary representation table of products in Z_{2^k} contains elements of $\xi_k^{(1)}$ and zero row as well as $W_k^{(\lambda)}$ rows and only the following rows:

$$H_{\alpha\left(2^{K+1}+2^{\lambda}\right)} = H_{\alpha 2^{K-1}} + H_{\alpha 2^{\lambda}}, \lambda = 0,1,\dots,2^{k-2}$$

We note that if H_α does not exist in $W_{k-1}^{(\lambda)}$, it is replaced by H_t, where $\alpha = t\left(\text{mod } 2^{k-1}\right)$.

Theorem 1

The set of sequences $W_k^{(\lambda)}, \quad \lambda = 2,\dots,2^{k-2}$ is equivalent to the set of sequences $W_k^{(1)}$.

Proof

We take

$$\eta_\alpha : W_k^{(1)} \to W_k^{(\lambda)}; \eta_\alpha\left(H_{2^i}\right) = H_{\alpha 2^i}, i = 0,\dots\dots\dots,k-1$$

where $\alpha = 2\lambda - 1$. η_α is injection and surjection and thereby is one-to-one correspondence (bijection) and its inverse is $\eta_\alpha^{-1} = \eta_{\alpha^{-1}}$.

Theorem 2

$\xi_k^{(\lambda)}$ is isomorphism to $\xi_k^{(1)}$.

Proof

Extend η_α to $\xi_k^{(1)}$ as follows.

$$H_{2^i}, H_{2^j} \in \xi_k^{(1)} \Rightarrow \eta_\alpha(H_{2^i} + H_{2^j}) = \eta_\alpha(H_{2^i}) + \eta_\alpha(H_{2^j})$$

where the addition is taken modulo 2.

As illustration, we take $k = 4$. Table 5 shows the products in Z_{2^4}, and Table 6 shows the binary representation for these products. We find

$$\omega_4^{(2)} = \begin{bmatrix} 0 & 3 & 6 & 9 & 12 & 15 & 2 & 5 & & 8 & 11 & 14 & 1 & 4 & 7 & 10 & 13 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 6 & 12 & 2 & 8 & 14 & 4 & 10 & \vdots & 0 & 6 & 12 & 2 & 8 & 1 & 14 & 10 \\ 0 & 12 & 8 & 14 & 0 & 12 & 8 & 4 & \vdots & 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & \vdots & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \end{bmatrix}$$

$$W_4^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0{:} & 0 & 1 & 1 & 0 & \vdots & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1{:} & 0 & 1{:} & 0 & 1 & 0 & 1 & \vdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\xi_4^{(2)} =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	r_0
0	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	r_1
0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	r_2
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	r_3
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	r_4
0	0	1	1	0	0	0	1	1	1	0	0	1	1	1	0	r_1+r_2
0	1	1	1	1	0	1	0	1	0	0	0	0	1	0	1	r_1+r_3
0	1	0	0	1	0	1	1	1	0	1	1	0	1	1	0	r_1+r_4
0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	r_2+r_3
0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	r_2+r_4
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	r_3+r_4
0	1	0	1	0	1	1	1	1	0	1	0	1	0	0	0	$r_1+r_2+r_3$
0	1	1	0	0	1	0	0	1	0	0	1	1	0	1	1	$r_1+r_2+r_4$
0	0	1	0	1	1	1	1	1	1	0	1	0	0	0	0	$r_1+r_3+r_4$
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	$r_2+r_3+r_4$
0	0	0	0	0	0	1	0	1	1	1	1	1	1	0	1	$r_1+r_2+r_3+r_4$

Similarly, we construct $\xi_4^{(3)}$ using

$$\omega_4^{(3)} = \begin{bmatrix} 0 & 5 & 10 & 15 & 4 & 9 & 14 & 3 & & 8 & 13 & 2 & 7 & 12 & 1 & 6 & 11 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 10 & 4 & 14 & 8 & 2 & 12 & 6 & \vdots & 0 & 10 & 4 & 14 & 8 & 2 & 12 & 6 \\ 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 & \vdots & 0 & 4 & 8 & 12 & 0 & 4 & 8 & 12 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & \vdots & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \end{bmatrix}$$

$$W_4^{(3)} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \vdots & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \vdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Similarly, we construct $\xi_4^{(4)}$ using

$$\omega_4^{(4)} = \begin{bmatrix} 0 & 7 & 14 & 5 & 12 & 3 & 10 & 1 & & 8 & 15 & 6 & 13 & 4 & 11 & 2 & 9 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & \vdots & 0 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 & \vdots & 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & \vdots & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \end{bmatrix}$$

$$W_4^{(4)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \vdots & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \vdots & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \vdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The sets $\xi_4^{(1)}$, $\xi_4^{(2)}$, $\xi_4^{(3)}$ and $\xi_4^{(4)}$ are different from each other in spite of existing common entries between them.

3.3 Third Step

(a) For $\alpha = 2\lambda - 1, \alpha > 2^{k-1} - 1$, each of the constructed sets $\xi_k^{(\lambda)}$ is identified with one of the sets $\xi_k^{(1)}, \xi_k^{(2))}, \ldots, \xi_k^{(2^k-2)}$. For example, when $\alpha = 9$, i.e. $\lambda = 5$, we find

$$\omega_4^{(5)} = \begin{bmatrix} 0 & 9 & 2 & 11 & 4 & 13 & 6 & 15 & & 8 & 1 & 10 & 3 & 12 & 5 & 14 & 7 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & \vdots & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 & \vdots & 0 & 12 & 8 & 4 & 0 & 12 & 8 & 4 \\ 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 & \vdots & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \end{bmatrix}$$

$$W_4^{(5)} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \vdots & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \vdots & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \vdots & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The rows of $W_4^{(5)}$ belong to $\xi_4^{(1)}$ and thereby $\xi_4^{(5)}$ and $\xi_4^{(1)}$ are identified. Similarly, $\xi_4^{(6)}$ and $\xi_4^{(2)}$ are identified, and so on.

(b) For α even, we get orthogonal subsets of the sets $\xi_k^{(1)}, \xi_k^{(2))}, \ldots, \xi_k^{(2^k-2)}$.

4. Results

(a) Suppose $Z_m = \{0,1,...,2^{k-1}-1\} \approx Z/mZ$ where $m = 2^k$ and $\chi : Z_m \to \{0,1\}$ such that $\chi(\{2^{k-1},...,2^k - 1\}) = 1$, $\chi(\{0,1,....,2^{k-1}-1\}) = 0$ and h_i is the row corresponding to $i \in Z_m$, $\chi(h_i) = H_i$ is the binary representation of h_i and $\omega_k^{(1)}$ is the matrix whose element are of Z_m where

$$\omega_k^{(1)} = \begin{bmatrix} h_{2^0} \\ h_{2^1} \\ \vdots \\ h_{2^{k-1}} \end{bmatrix} = \begin{bmatrix} & h_1 & \\ \cdots & \cdots & \cdots \\ 2\omega_{k-1}^{(1)} & \vdots & 2\omega_{k-1}^{(1)} \end{bmatrix}; \quad k \geq 2 \quad ; \omega_1^{(1)} = [0 \quad 1].$$

Suppose $\chi(\omega_k^{(1)}) = W_K^{(1)}$. Then, the matrix

$$W_k^{(1)} = \begin{bmatrix} H_1 \\ H_{2^1} \\ H_{2^2} \\ \vdots \\ H_{2^{k-1}} \end{bmatrix} = \begin{bmatrix} & H_1 & \\ \cdots & \cdots & \cdots \\ W_{k-1}^{(1)} & \vdots & W_{k-1}^{(1)} \end{bmatrix}, \quad k \geq 2, \quad W_k^{(1)} = [0 \quad 1]$$

generates Walsh sequences $\xi_k^{(1)}$ of order 2^k. The rows of $W_k^{(1)}$ are identified in order on R_1, R_2, R_3, R_4, Rademacher sequences of order *k*.

(b) The binary representation of the multiplication table on Z_m contains, in addition to $H_0 = R_0$, the sequences

$$H_{2^{k-1}+2^\lambda} = H_{2^{k-1}} + H_{2^\lambda} \qquad , \lambda = 0,...,2^{k-2}.$$

The number of Walsh sequences in the table is 2^k.

(c) Suppose $\varphi_\alpha : Z_m \to Z_m, \varphi_\alpha(x) = \alpha x, \alpha = 2\lambda - 1 \in Z_m, 1 < \alpha \leq 2^{k-1} - 1$
i.e., $\lambda = 2,3,\ldots,2^{k-2}$ and $\varphi_\alpha(\omega_k^{(1)}) = \omega_k^\lambda$. Then

$$\omega_k^{(\lambda)} = \begin{bmatrix} h_\alpha \\ h_{\alpha 2^1} \\ \vdots \\ h_{\alpha 2^{k-1}} \end{bmatrix} = \begin{bmatrix} & h_\alpha & \\ \cdots & \cdots & \cdots \\ 2\alpha\omega_{k-1}^{(1)} & \vdots & 2\alpha\omega_{k-1}^{(1)} \end{bmatrix} = \begin{bmatrix} & h_\alpha & \\ \cdots & \cdots & \cdots \\ 2\omega_{k-1}^{(\lambda)} & \vdots & 2\omega_{k-1}^{(\lambda)} \end{bmatrix}, k \geq 2, \quad \lambda = 2,3,...,2^{2k-2}$$

Suppose $\chi(\omega_k^{(\lambda)}) = W_k^{(\lambda)}$. Then

$$W_k^{(\lambda)} = \begin{bmatrix} H_\alpha \\ H_{\alpha 2^1} \\ \vdots \\ H_{\alpha 2^{k-1}} \end{bmatrix} = \begin{bmatrix} H_\alpha \\ \cdots\cdots\cdots\cdots \\ W_{k-1}^{(\lambda)} \; \vdots \; W_{k-1}^{(\lambda)} \end{bmatrix}, \quad k \geq 2, \quad \lambda = 2,3,\ldots,2^{2k-2}$$

where $W_1^{(\lambda)} = [0 \quad 1]$. φ_α is one-to-one correspondence (bijection) between rows of $\omega_k^{(1)}$ and rows of $\omega_k^{(\lambda)}$; and $\eta_\alpha : H_i \rightarrow H_{\alpha i}$ is bijection between $\omega_k^{(1)}$ and $W_1^{(\lambda)}$.

The additive group $\xi_k^{(\lambda)}$ generated from $W_k^{(\lambda)}$ is isomorphism to $\xi_k^{(1)}$.

(d) The binary representation of the multiplication table contains, in addition to the rows of $W_k^{(\lambda)}$ and the zero row, only the following rows from $\xi_k^{(\lambda)}$

$$H_{\alpha 2^{k-1} + \alpha 2^\lambda} = H_{\alpha 2^{k-1}} + H_{\alpha 2^\lambda}, \quad \lambda = 0,\ldots,2^k - 2.$$

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